Hints and outlines to Solutions of Certain Problems in H.W. #3

#8 P67

Equation (3) before Thm 3.2.4 ~~Aff~~ requires a fixed number of factors

9 P67

$$y_n = \sqrt{n+1} - \sqrt{n} = \left(\sqrt{n+1} - \sqrt{n}\right)\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1}+\sqrt{n}}$$

$$\sqrt{n}\, y_n = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}$$

#10 a $\left(3\sqrt{n}\right)^{\frac{1}{2n}} = \left(\sqrt{3}\right)^{\frac{1}{n}}\left(n^{\frac{1}{4n}}\right) = \left(\sqrt{3}\right)^{\frac{1}{n}}\left[\left(n^{\frac{1}{n}}\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}$ and use established results and theorems.

14) $b^n < (a^n + b^n) < 2b^n$

$\therefore\ b < (a^n + b^n)^{\frac{1}{n}} < (2)^{\frac{1}{n}} b$ and apply squeeze theorem.

P 74

#3 $x_1 \geq 2$. Suppose $x_k \geq 2$, then

$$x_{k+1} = 1 + \sqrt{x_k - 1} \geq 1 + \sqrt{2-1} = 2$$

so by induction $x_n \geq 2$ for all $n \in \mathbb{N}$.

$$x_{k+1} - x_k = 1 + \sqrt{x_k - 1} - x_k = \sqrt{x_k - 1} - (x_k - 1) = \sqrt{x_k - 1}\left(1 - \sqrt{x_k - 1}\right)$$

since $x_k \geq 2$ second factor is 0 or negative. $\therefore$ Sequence is decreasing

To find limit, let $k \to \infty$.

#11 $y_{n+1} - y_n = \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{1}{2n+1} - \frac{1}{2(n+1)} > 0$

$$y_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \underbrace{\frac{1}{n+1} + \cdots + \frac{1}{n+1}}_{n \text{ summands}} = \frac{n}{n+1} \le 1$$

$\therefore$

#13 ~~6~~

$$\left(1+\frac{1}{n+1}\right)^n = \left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1}$$

P80

#6)

(a) $x_{n+1} < x_n \Leftrightarrow (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \Leftrightarrow n^{\frac{1}{n+1}}\left(1+\frac{1}{n}\right)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \Leftrightarrow n\left(1+\frac{1}{n}\right) < n^{1+\frac{1}{n}}$

$$\Leftrightarrow \left(1+\frac{1}{n}\right) < n^{\frac{1}{n}} \Leftrightarrow \left(1+\frac{1}{n}\right)^n < n.$$

Inequality $\left(1+\frac{1}{n}\right)^n < n$ True for $n=3$.

Suppose true for k. Then.

$$\left(1+\frac{1}{k+1}\right)^{k+1} < \left(1+\frac{1}{k}\right)^{k+1} = \left(1+\frac{1}{k}\right)^k\left(1+\frac{1}{k}\right) \underset{\substack{\uparrow \\ \text{inductive hyp}}}{<} k\left(1+\frac{1}{k}\right) = k+1.$$

(b) Subsequence $(2n)^{\frac{1}{2n}}$ converges to same limit, say x as does $n^{\frac{1}{n}}$

Now: $(2n)^{\frac{1}{2n}} = (\sqrt{2})^{\frac{1}{n}}\left[\left(n^{\frac{1}{n}}\right)\right]^{\frac{1}{2}} \to (1)(x^{\frac{1}{2}})$

$\therefore$ $x = x^{\frac{1}{2}}$ and $x=0$ or $x=1$ but since $x_n \ge 1$, $x=1$

7(c) $\left(1+\frac{1}{n^2}\right)^{2n^2} = \left[\left(\frac{1+\frac{1}{n^2}}{1}\right)^{n^2}\right]^2$ Since $\{n^2\}$ is a subsequence of $\{n\}$

$\left(1+\frac{1}{n^2}\right)^{n^2} \to e$ and $\left(1+\frac{1}{n^2}\right)^{2n^2} \to e^2$