Partial Solutions to H.W #4

P 86

2a

$$|X_n - X_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{1}{n} - \frac{1}{m}\right| < \frac{1}{n} + \frac{1}{m}$$

Let $\varepsilon > 0$ be given. Then by Archimedean property, $\exists\, K(\varepsilon)$ s.t. $\frac{1}{K(\varepsilon)} < \frac{\varepsilon}{2}$. $\therefore$ For $n, m > K(\varepsilon)$, $|X_n - X_m| < \varepsilon$

(2b)

Let $m > n$

$|X_m - X_n| = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \dots + \frac{1}{m!}$

Since $2^{k-1} < k!$ for $k > 3$ (Proved in text or by induction)

we have

$$|X_m - X_n| \le \frac{1}{2^n} + \frac{1}{2^{n+1}} + \dots + \frac{1}{2^m} = \frac{1}{2^n}\left[1 + \frac{1}{2} + \dots + \left(\frac{1}{2}\right)^{m-n}\right]$$

$$= \frac{1}{2^n}\left[\frac{1 - \left(\frac{1}{2}\right)^{m-n-1}}{1 - \frac{1}{2}}\right] \le \frac{1}{2^{n-1}}$$

3(b) Let n be odd, let $m = 2n+1$. Then

$$|X_m - X_n| = \left|(m-n) - \frac{1}{m} - \frac{1}{n}\right| = (m-n) - \frac{1}{m} - \frac{1}{n} > (n+1) > 1$$

for $n > 5$, say

Hence for each $k > 5$, take $n = 2k+1$, $m = 2(2k+1)$

and $|X_m - X_n| > 1$.

(4)

$$|(X_m + Y_m) - (X_n + Y_n)| \le |X_m - X_n| + |Y_m - Y_n|$$

$$|X_m Y_m - X_n Y_n| = |X_m Y_m - X_m Y_n + X_m Y_n - X_n Y_n|$$

$$\le |X_m|\,|Y_m - Y_n| + |Y_n|\,|X_m - X_n|$$

$$\le A|Y_m - Y_n| + B|X_m - X_n|$$

Since Cauchy sequences are bounded

#12

$$|x_{n+1} - x_n| = \left| \frac{1}{2+x_n} - \frac{1}{2+x_{n-1}} \right| = \left| \frac{(x_n - x_{n-1})}{(2+x_n)(2+x_{n-1})} \right|$$

Now, since $x_1 > 0$ it follows by inductive argument + defining recursion that $x_n > 0$ for all n, Hence

$$|x_{n+1} - x_n| \leq \frac{|x_n - x_{n-1}|}{(2)(2)} = \frac{1}{4} |x_n - x_{n-1}|$$

Since (x_n) is contractive it is convergent. Hence

Letting $n \to \infty$ in

$$x_{n+1} = \frac{1}{2+x_n}$$

$$x = \frac{1}{2+x} \quad \text{or} \quad x^2 + 2x - 1 = 0$$

$$\therefore x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 + \sqrt{2}$$

Take + sign since $x \geq 0$ (since $x_n > 0$)

P 88

#2

(a) $x_n = n$
$y_n = n$

(b) $x_n = n^2$
$y_n = n.$