Partial Solutions to H.W. #5

P 104 # 7

$$|x^3-c^3| = |x-c||x^2+cx+c^2| \le |x-c|\left[|x|^2+|cx|+c^2\right] \quad (1)$$

Now $|x| \le |x-c|+|c|$. Let $\delta_1 = 1$ and take $|x-c|<1$. Then

$$|x| < 1+|c|$$

Substituting into (1) gives

$$|x^3-c^3| \le |x-c|\left[(1+|c|)^2+|c|(1+|c|)+|c|^2\right]$$

The quantity in square brackets is a constant, call it K & we have

$$|x^3-c^3| \le |x-c|K.$$

Let $\varepsilon>0$ be given. Let $\delta = \min\left(1, \frac{\varepsilon}{K}\right)$. Then for $0<|x-c|<\delta$,

$$|x^3-c^3| < \varepsilon$$

P 104

#10 a

$$|x^2+4x-12| = |x+6||x-2|$$

Proceed as in #7 to bound $|x+6|$ & get

$$|(x^2+4x)-12| \le K|x-2|$$

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#10b $\left|\frac{x+5}{2x+3}-4\right| = \left|\frac{x+5-8x-12}{2x+3}\right| = \left|\frac{7(x+1)}{2x+3}\right| = 7\frac{|x-(-1)|}{|2x+3|}$

Now let $\delta_1 = \frac{1}{4}$ and take $|x+1|<\frac{1}{4}$. Then

$2x+3$ is >0 and is smallest when $x=-\frac{5}{4}$

When ~~x=5~~ $x=-5/4$ we get $2\left(-\frac{5}{4}\right)+3 = \frac{1}{2}$

Hence: for $|x+1|<1/4$ we have

$$\left|\frac{x+5}{2x+3}-4\right| < 14|x+1|$$

Let $\varepsilon>0$ be given. Let $\delta = \min\left(\frac{1}{4}, \frac{\varepsilon}{14}\right)$. Then for

$|x+1|<\delta$, we get that

$$\left|\frac{x+5}{2x+3}-4\right| < \varepsilon$$

INSERT $\longrightarrow$ See Next Page for #11C

P110 3) $$\frac{\sqrt{1+2x}-\sqrt{1+3x}}{x+2x^2} = \frac{\left(\sqrt{1+2x}+\sqrt{1+3x}\right)}{(x+2x^2)}\frac{\left(\sqrt{1+2x}-\sqrt{1+3x}\right)}{\sqrt{1+2x}+\sqrt{1+3x}}$$

Do algebra and use Theorems.

P110 4) $$-1 \le \cos\frac{1}{x} \le 1$$

For $x>0$

$-x \le x\cos\frac{1}{x} \le x$ or $-|x| \le x\cos\frac{1}{x} \le |x|$

For $x<0$

$-x \ge x\cos\frac{1}{x} \ge x$ or $|x| \ge x\cos\frac{1}{x} \ge -|x|$

$\therefore$ For all $x \ne 0$

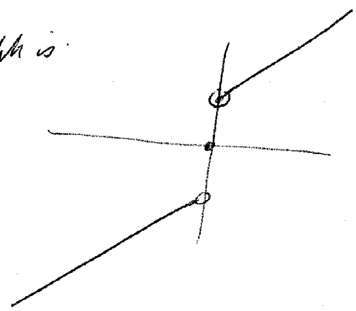
$$-|x| \le x\cos\frac{1}{x} \le |x|$$

As $x\to 0$, $|x|\to 0$ + so by squeeze theorem $x\cos\frac{1}{x}\to 0$.

#11c

$f(x) = x + \operatorname{sgn} x$

Graph is:



Let $x_n = \frac{(-1)^n}{n}$ then $\lim_{n \to \infty} x_n = 0$

Consider subsequence $\{x_{2n}\}$. Then $f(x_{2n}) = 1$ $n = 1, 2, 3 \ldots$

Consider " $\{x_{2n+1}\}$ Then $f(x_{2n+1}) = -1$ $n = 1, 2, 3, \ldots$

$\therefore$ Sequence $\{f(x_n)\}$ does not converge

$\therefore$ $\lim_{x \to 0} f(x)$ does not exist, for if it did exist, and were equal to L, then every sequence $\{x_n\}$ with $x_n \neq 0$ and $\lim_{n \to \infty} x_n = 0$ would also have the property that $\lim_{n \to \infty} f(x_n) = L$.

#4) Let $L = \lim_{x \to c} f(x)$

Case 1 $L > 0$

Then

$$\left|\sqrt{f(x)} - \sqrt{L}\right| = \frac{|f(x) - L|}{\left|\sqrt{f(x)} + L\right|} = \frac{|f(x) - L|}{\sqrt{f(x)} + L} \le \frac{|f(x) - L|}{L}$$

$\uparrow$ since $L > 0$ $f(x) \geq 0$

Case 2 $L = 0$

$$\left|\sqrt{f(x)} - 0\right| = \sqrt{f(x)}$$

Let $\varepsilon > 0$ be given then $\lim_{x \to c} f(x) = 0 \Rightarrow \exists \delta > 0$

s.t. for $x > 0$, $0 < |x - c| < \delta$, $[f(x)] < \varepsilon^2$.

P124 # 11

Let x be a rational number. By the density of the rationals, $\exists$ a sequence r_n of rationals such that $\lim r_n = x$. Since f is continuous $\lim_{n \to \infty} f(r_n) = f(x)$. But $f(r_n) = 0$ for each n, so

$f(x) = 0$