Partial Solutions to H.W. #6

P129 #1 Let $f(x) = \begin{matrix} 1 \text{ if } x \text{ is rational (or irrational)} \\ -1 \text{ if } x \text{ is irrational (or rational)} \end{matrix}$

Then f is discontinuous at all points. On the other hand $|f|(x) \equiv |f(x)| = 1$ and is continuous everywhere

P135: #1 Since f is continuous on the closed bounded interval $[a,b]$, there exists a point c in $[a,b]$ such that $f(x) \geq f(c)$ for all x in $[a,b]$. By hypothesis $f(c) > 0$. Take $\alpha = f(c)$.

#3) Choose $x_1 \in I$. By hypothesis, $\exists\, x_2 \in I$ such that $|f(x_2)| \leq \frac{1}{2}|f(x_1)|$. By hypothesis $\exists\, x_3 \in I$ such that

$$|f(x_3)| \leq \frac{1}{2}|f(x_2)| \leq \frac{1}{4}|f(x_1)|$$

Thus, suppose $x_1, \ldots, x_k$ have been chosen so that

$$|f(x_j)| \leq \frac{1}{2^{j-1}}|f(x_1)|, \quad j = 2, \ldots, k \qquad (0)$$

By hypothesis, $\exists\, x_{k+1} \in I$ such that

$$|f(x_{k+1})| \leq \frac{1}{2}|f(x_k)| \leq \frac{1}{2^k}|f(x_1)| \qquad (1)$$

by inductive hypothesis (0)

Thus we have inductively defined a sequence of points $\{x_k\}$ such that (1) holds. It follows from (1) that

$$\lim_{k\to\infty} |f(x_{k+1})| = 0$$

Hence $\lim_{k\to\infty} f(x_k) = 0. \qquad (2)$

Since $I = [a, b]$ a closed bounded interval by such that $\lim_{j \to \infty} x_{k_j} = c$, where c is a point in I

From (2) we have that
$$\lim_{j \to \infty} f(x_{k_j}) = 0$$

But since f is continuous,
$$\lim_{j \to \infty} f(x_{k_j}) = f(c)$$

Hence $f(c) = 0$

~~# 7 $x = \cos x$ has a solution~~

\# 4) Let $p(x)$ be a polynomial of odd degree. Then
$$\lim_{x \to +\infty} p(x) = +\infty \qquad \lim_{x \to -\infty} p(x) = -\infty$$
Hence $\exists$ points $a < b$ such that $f(a) < 0 < f(b)$.
By the Location of roots theorem, $\exists$ a point c in (a, b) such that $f(c) = 0$. (Of course we use the fact that a polynomial is continuous on $(-\infty, \infty)$).

\# 7) $x = \cos x$ has a solution $\Leftrightarrow f(x) = x - \cos x$ has a zero on $[0, \pi/2]$ on $[0, \pi/2]$

But $f(0) = -1$ $\quad f(\pi/2) = \pi/2$. Hence by location of roots theorem f has a solution ~~on~~ in $(0, \pi/2)$