MA 341

H.W. #7 Partial Sol'ns to H.W. Problems

#2) P. 144

(a) To show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[1, \infty)$ we must show that for each $\varepsilon > 0$, $\exists \delta(\varepsilon) > 0$ such that for all x, u in $[1, \infty)$ and $|x-u| < \delta(\varepsilon)$, then $\left|\frac{1}{x^2} - \frac{1}{u^2}\right| < \varepsilon$.

Now:

$$\left|\frac{1}{x^2} - \frac{1}{u^2}\right| = \frac{|x+u||x-u|}{|x^2u^2|} \underset{\substack{\uparrow \\ x>0,\ u>0}}{=} \left[\frac{x}{x^2u^2} + \frac{u}{x^2u^2}\right]|x-u|$$

$$= \left[\frac{1}{xu^2} + \frac{1}{x^2u}\right]|x-u| \underset{\substack{\uparrow \\ x \geq 1,\ u \geq 1}}{\leq} 2|x-u|$$

Let $\delta(\varepsilon) = \varepsilon/2$, then for $|x-u| < \delta(\varepsilon)$, we have

$$\left|\frac{1}{x^2} - \frac{1}{u^2}\right| < \varepsilon$$

(b) We must show that $\exists \varepsilon_0 > 0$ such that for each $k > 0$, $\exists x_k, u_k$ in $(0, \infty)$ such that $|x_k - u_k| < 1/k$ and $\left|\frac{1}{(x_k)^2} - \frac{1}{(u_k)^2}\right| \geq \varepsilon_0$

For each k, let $x_k = \frac{1}{k}$ $u_k = \frac{1}{k+1}$. Then

$$|x_k - u_k| = \left|\frac{1}{k} - \frac{1}{k+1}\right| = \frac{1}{k(k+1)} < \frac{1}{k}$$

But

$$\left|\frac{1}{(x_k)^2} - \frac{1}{(u_k)^2}\right| = \left|k^2 - (k+1)^2\right| = 2k+1 \geq 1$$

Let $\varepsilon_0 = 1$.

#6) Must show that for each $\varepsilon > 0$, $\exists$ a $\delta(\varepsilon)$ such that if $|x-u| < \delta(\varepsilon)$,

then $|(fg)(x) - (fg)(u)| = |f(x)g(x) - f(u)g(u)| < \varepsilon$

Now:

$$|f(x)g(x) - f(u)g(u)| \leq |f(x)g(x) - f(x)g(u)| + |f(x)g(u) - f(u)g(u)|$$

$$= |f(x)||g(x) - g(u)| + |g(u)||f(x) - f(u)|$$

$$\underset{\substack{\uparrow \\ f \text{ bounded, by } K \text{, say}}}{\leq} K|g(x) - g(u)| + L|f(x) - f(u)|$$

g bounded by L say.

f unif cont $\Rightarrow \exists \delta_1$ s.t. for $|x-u| < \delta_1$, $|f(x) - f(u)| < \varepsilon/2L$

g " " " $\exists \delta_2$ " " " $< \delta_2$ $|g(x) - g(u)| < \frac{\varepsilon}{2K}$

$\therefore$ For $|x-u| < \delta \equiv \min(\delta_1, \delta_2)$, $|fg(x) - fg(u)| < \varepsilon$.

9) $$\left|\frac{1}{f(x)} - \frac{1}{f(u)}\right| = \left|\frac{f(x) - f(u)}{f(x)\,f(u)}\right| = \frac{|f(x) - f(u)|}{|f(x)||f(u)|}$$

Now $|f(x)| \geq k > \Rightarrow \left|\frac{1}{f(x)}\right| \leq \frac{1}{k}$ $\left|\frac{1}{f(u)}\right| \leq \frac{1}{k}$, so

$$\left|\frac{1}{f(x)} - \frac{1}{f(u)}\right| \leq \frac{1}{k^2}|f(x) - f(u)|$$

f unif cont $\Rightarrow \exists\, \delta(\varepsilon)$ such that for $|x-u| < \delta$,

$$|f(x) - f(u)| < \varepsilon k^2$$

+ so $\frac{1}{f}$ is uniformly continuous

12) f unif cont. on $[a,\infty)$ $a>0$ $\Rightarrow$ For each $\varepsilon>0$, $\exists\delta_1(\varepsilon)$ such that (1)
for x, u in $[a,\infty)$ and $|x-u|<\delta_1$ $|f(x)-f(u)|<\varepsilon/2$

f cont on $[0,\infty)$ $\Rightarrow$ f continuous on $[0,a]$, & so f is
uniformly continuous on $[0,a]$
i.e For each $\varepsilon>0$, $\exists\delta_2(\varepsilon)$ such that for $x,u\in[0,a]$ (2)
and $|x-u|<\delta_2$, $|f(x)-f(u)|<\varepsilon/2$

Now let $x\in[0,a]$, $u\in[a,\infty)$ and let

$$|x-u|<\delta\equiv\min(\delta_1,\delta_2).$$

Then $|f(x)-f(u)| \le |f(x)-f(a)|+|f(a)-f(u)|$

Now $|x-u|<\delta$, $x\in[0,a]$, $u\in[a,\infty)$ $\Rightarrow$ $|x-a|<\delta$
$|u-a|<\delta$

Since $x, a\in[0,a]$ we have for $|x-u|<\delta$

$$|f(x)-f(a)|<\varepsilon/2$$

Since $u, a\in[a,\infty)$ we have

$$|f(u)-f(a)|<\varepsilon/2$$

& ~~So~~ for $x\in[0,a]$, $u\in[a,\infty)$ and $|x-u|<\delta$, we
have

$$|f(x)-f(u)|<\varepsilon$$

Combining this with (1) & (2), we have (since $\varepsilon/2<\varepsilon$) that
whenever $|x-u|<\delta$, then $|f(x)-f(u)|<\varepsilon$

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These Problems are 1st semester calculus problems.

P 155 #4

f increasing $\Rightarrow$ If $x_2 > x_1$ then $f(x_2) \geqslant f(x_1)$

g " $\Rightarrow$ " " " $g(x_2) \geqslant g(x_1)$

Since f and g are positive, for $x_2 > x_1$

$$f(x_2)g(x_2) \geqslant f(x_1)g(x_1)$$

P 175 #5

$$f(x) = x^{\frac{1}{n}} - (x-1)^{\frac{1}{n}} \qquad x \geqslant 1$$

$$f'(x) = \frac{1}{n}\left[x^{\frac{1}{n}-1} - (x-1)^{\frac{1}{n}-1}\right] = \frac{1}{n}\left[\frac{1}{x^{1-\frac{1}{n}}} - \frac{1}{(x-1)^{1-\frac{1}{n}}}\right]$$

For ~~Since~~ $x > 1$, $x > x-1 > 0$

Therefore $\frac{1}{x-1} > \frac{1}{x}$ and $\frac{1}{(x-1)^{1-\frac{1}{n}}} > \frac{1}{x^{1-\frac{1}{n}}}$

$\therefore$ $f'(x) < 0$ & so f is decreasing.

$\therefore$ If $a > b$, $\left(\frac{a}{b}\right) > 1$ and

$$f\left(\frac{a}{b}\right) < f(1)$$

or $\left(\frac{a}{b}\right)^{\frac{1}{n}} - \left(\frac{a}{b} - 1\right)^{\frac{1}{n}} < 1$

$\therefore$ $a^{\frac{1}{n}} - (a-b)^{\frac{1}{n}} < b^{\frac{1}{n}}$

or $a^{\frac{1}{n}} - b^{\frac{1}{n}} < (a-b)^{\frac{1}{n}}$