Outline of Solutions for H.W #8 MA341

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If $x=0$, then for each $n \in \mathbb{N}$, $\frac{nx}{1+n^2x^2}=0$. Hence,

for $x=0$

$$\lim_{n\to\infty} \frac{nx}{1+n^2x^2}=0$$

If $x \neq 0$

$$\left|\frac{nx}{1+n^2x^2}\right| = \left|\frac{x}{x^2}\,\frac{n}{(\frac{1}{x^2}+n^2)}\right| = \frac{1}{|x|}\,\frac{n}{(n^2+\frac{1}{x^2})} \le \frac{1}{|x|}\,\frac{n}{n^2} = \frac{1}{n|x|} \quad (1)$$

Hence by Squeeze Thm

$$\lim_{n\to\infty}\left|\frac{nx}{1+n^2x^2}\right| = 0 \quad \& \quad \lim_{n\to\infty}\frac{nx}{1+n^2x^2}=0$$

From (1) we see that for $x \in [a,\infty)$

$$\left|\frac{nx}{1+n^2x^2}\right| \le \frac{1}{n|a|}$$ ∴ so the convergence is uniform on $[a,\infty)$

To show non-uniform convergence on $[0,\infty)$, we must show that $\exists \varepsilon_0$ such that for each $k \in \mathbb{N}$, $\exists n_k \ge k$ and a point x_{n_k} in $[0,\infty$ such that $|f_{n_k}(x_{n_k})-0| = |f(x_{n_k})| > \varepsilon_0$. (where $f_n(x) = \frac{nx}{1+n^2x^2}$)

For each k, let $n_k = k$ and $x_{n_k} = \frac{1}{k}$. Then

$$f_{n_k}(x_{n_k}) = \frac{k(\frac{1}{k})}{1+k^2(\frac{1}{k^2})} = \frac{1}{2}$$

Take $\varepsilon_0 = \frac{1}{4}$

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Let $f_n(x) = \frac{nx}{1+nx}$

Then: $f_n(0) = 0$ for all n, so $\lim_{n\to\infty} f_n(0) = 0$

Now let $x > 0$. Then

$$f_n(x) = \frac{nx}{nx} \frac{1}{(1+\frac{1}{nx})} = \frac{1}{1+\frac{1}{nx}}$$

$\therefore$ By standard Theorems $\lim_{n\to\infty} f_n(x) = 1$ if $x > 0$.

On $[a,\infty)$ convergence is uniform

$$|f_n(x) - 1| = \left|\frac{nx}{1+nx} - 1\right| = \left|\frac{-1}{1+nx}\right| = \frac{1}{1+nx} \le \frac{1}{1+na},$$

whence uniform convergence follows.

Non uniform convergence on $[0,\infty)$

For each k, let $n_k = k$, let $x_k = \frac{1}{k}$.

Then

$$\left|f_k\left(\tfrac{1}{k}\right) - 1\right| = \frac{1}{1+k\left(\frac{1}{k}\right)} = \frac{1}{2}$$

Take $\varepsilon_0 = \frac{1}{4}$.

(4, 14)

Let $f_n(x) = \dfrac{x^n}{1+x^n}$ $\quad f_n(0) = 0 \quad \therefore \lim f_n(0) = 0$

$f_n(1) = \frac{1}{2} \quad \therefore \lim f_n(1) = \frac{1}{2}$

If $0 < x < 1$, then $\lim\limits_{n\to\infty}(x^n) = 0$ and so $\lim\limits_{n\to\infty} f_n(x) = 0$

If $x > 1$ then $\lim\limits_{n\to\infty} x^n = +\infty$.

Now $f_n(x) = \dfrac{x^n}{x^n\left(1+\frac{1}{x^n}\right)} = \dfrac{1}{\left(1+\frac{1}{x^n}\right)}$

$\therefore$ for $x > 1$, $\lim\limits_{n\to\infty} f_n(x) = 1$

Convergence is uniform on $[0,b]$ if $b<1$.

$|f_n(x) - 0| = \dfrac{x^n}{1+x^n} < x^n < b^n$, from which uniform convergence follows.

To show non-uniform convergence on $[0,1]$,

For each $k \geq 2$ take $n_k = k$ and

$$x_k = \left(1-\frac{1}{k}\right)^{\frac{1}{k}}$$

Then $|f_k(x_k) - 0| = |f_k(x_k)| = f_k(x_k) = \dfrac{\left(1-\frac{1}{k}\right)}{1+1-\frac{1}{k}}$

$$= \frac{1-\frac{1}{k}}{2-\frac{1}{k}} > \frac{1}{2}$$

Take $\varepsilon_0 = \frac{1}{4}$.