SOLUTION.

MA 387 Exam 1 September 19, 2002

Name______________________________ Stud. No.____________________

1. (15) Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$. Find each of the following sets.

(a) $A \cap \overline{B}$. $= \{1, 2\}$

(b) $\mathcal{P}(A)$. $= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\ \{1,2,3\}\}$

(c) $A \times B$. $= \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle\ \langle 3,3\rangle, \langle 3,4\rangle\}$

2. (10) Prove that for all sets A, B, and C, if $B \subseteq C$ then $A - C \subseteq A - B$.

proof: Suppose $B \subseteq C$.

$x \in A - C \rightarrow x \in A \wedge x \notin C$ (Def of $A-C$)

$\rightarrow x \in A \wedge x \notin B$ (because $B \subseteq C$)

$\rightarrow x \in A - B$ (Def of $A - B$)

$\therefore A - C \subseteq A - B$

3. (15) Prove that for all sets A and B,

$$(A - \overline{B}) \cup (\overline{B} - A) = \overline{(\overline{A} \cap B)} - \overline{(\overline{A} \cup B)}.$$

R.S. $= (A \cup \overline{B}) \cap (\overline{A} \cup B)$ [De Morgan's Rule & the def of $-$ & the Rule of double negation].

$= (A \cap \overline{A}) \cup (A \cap B) \cup (\overline{B} \cap \overline{A}) \cup (\overline{B} \cap B)$ [Distributive Rule]

$= (A \cap B) \cup (\overline{B} \cap \overline{A})$ [$A \cap \overline{A} = \overline{B} \cap B = \emptyset$].

$= (A - \overline{B}) \cup (\overline{B} - \overline{\overline{A}})$ [Def of $-$]

$= (A - \overline{B}) \cup (\overline{B} - A)$ [Rule of Double negation]

$=$ L.S.

4. (10) Show that the following statement is NOT a theorem of set theory: For all sets A, B, and C, if $B \subseteq \overline{C \cup A}$ and $C \subseteq \overline{B \cup A}$ then $C = \emptyset$.

Let $A = \{1\}$, $B = \{2\}$, & $C = \{3\}$.

$C \cup A = \{1, 3\}$ & $B \subseteq \overline{C \cup A}$

$B \cup A = \{1, 2\}$ & $C \subseteq \overline{B \cup A}$

but $C \neq \emptyset$

5. (10) If $f(x) = 7\sqrt{x^2+1}$, express f as the composition of four functions, none of which is the identity function.

Let $f_1(x) = 7x$, $f_2(x) = \sqrt{x}$, $f_3(x) = x+1$, $f_4(x) = x^2$

Then $f(x) = f_1 \circ f_2 \circ f_3 \circ f_4(x)$

6. (15) Show that if R is an equivalence relation, then $R \circ R^{-1} = R$.

$\langle x,y\rangle \in R \circ R^{-1}$ iff $(\exists z)(\langle x,z\rangle \in R^{-1} \wedge \langle z,y\rangle \in R)$ Def. of $\circ$

iff $(\exists z)(\langle z,x\rangle \in R \wedge \langle z,y\rangle \in R)$ Def of R^{-1}

iff $(\exists z)(\langle x,z\rangle \in R \wedge \langle z,y\rangle \in R)$ R is symmetric

$\rightarrow$ $\langle x,y\rangle \in R$ R is transitive.

If $\langle x,y\rangle \in R$ Then $\langle y,y\rangle \in R$ because R is reflexive.

$\therefore$ $\langle y,x\rangle \in R$ because R is symmetric

$\therefore$ $\langle x,y\rangle \in R^{-1}$ by the definition of R^{-1}.

$\therefore$ $\langle x,y\rangle \in R^{-1}$ and $\langle y,y\rangle \in R$, so by the definition of $\circ$, $\langle x,y\rangle \in R \circ R^{-1}$.

$\therefore$ $R \circ R^{-1} = R$

7. (15) Prove that for all sets A and B, $\bigcap\bigcup\langle A,B\rangle = A\cap B$.

$\langle A, B\rangle = \{\{A\}, \{A, B\}\}$.

$\bigcup\langle A, B\rangle = \{A\} \cup \{A, B\} = \{A, B\}$.

$\bigcap\bigcup\langle A, B\rangle = \bigcap\{A, B\} = A \cap B$.

8. (10) Suppose that A is a set of functions that is linearly ordered by the subset relation, $\subseteq$, prove that $\bigcup A$ is a function.

$\bigcup A$ is a set of ordered pairs.

Suppose $\langle x_1, y\rangle, \langle x_2, y\rangle \in \bigcup A$. (show $x_1 = x_2$)

Then $(\exists f_1 \in A)(\langle x_1, y\rangle \in f_1)$ and

$(\exists f_2 \in A)(\langle x_2, y\rangle \in f_2)$.

Since A is linearly ordered by $\subseteq$, either

(1) $f_1 \subseteq f_2$ or (2) $f_2 \subseteq f_1$.

If (1) holds then $\langle x_1, y\rangle$ & $\langle x_2, y\rangle$ both belong to f_2 which contradicts the fact that f_2 is a function if $x_1 \neq x_2$.

The proof is similar if (2) holds.