SOLUTION

MA 387 | Exam I | February 26, 1975

Name ______________________________

1. Prove that for all sets A and B, $A \subseteq B$ if and only if $A \cup (B - A) = B$.

proof:

$\rightarrow$ Suppose $A \subseteq B$ then

$$\begin{aligned} A \cup (B - A) &= A \cup (B \cap \bar{A}) \\ &= (A \cup B) \cap (A \cup \bar{A}) \\ &= (A \cup B) \cap U \\ &= A \cup B \\ &= B \quad \text{because } A \subseteq B \end{aligned}$$

$\leftarrow$ Suppose $A \cup (B - A) = B$ and $x \in A$ then since $x \in A \cup (B - A)$, it follows that $x \in B$. Therefore, $A \subseteq B$

2. Show that if $F : A \to B$, $G : B \to A$ and $F \circ G = i_B$ then F maps A onto B and G is $1-1$.

(a) F maps A onto B.

Suppose $y \in B$. Then since $F \circ G = i_B$, $F(G(y)) = y$. Since $G : B \longrightarrow A$, $G(y) \in A$. Therefore $x = G(y)$ is the element in A which is mapped onto y by the function F.

(b) G is 1-1.

Suppose $x, y \in B$ and $G(x) = G(y)$. Then $F(G(x)) = F(G(y))$. But, by hypotheses, $F \circ G = i_B$, so $F(G(x)) = x$ and $F(G(y)) = y$. Consequently, $x = y$ so G is 1-1

3. Calculate each of the following:

(a) $\bigcap_{n \in z^+} [0,1/n] = \{0\}$

proof:

Clearly $0 \in [0, 1/m]$ for every $m > 0$, so $\{0\} \subseteq \bigcap_{m \in z^+} [0, 1/m]$

Conversely if $x \in \bigcap_{m \in z^+} [0, \frac{1}{m}]$ then $x \in [0, \frac{1}{m}]$ for every $m \in z^+$. This implies $x \geq 0$. If $x > 0$ choose $m \in z^+$ so that $0 < \frac{1}{m} < x$. Then $x \notin [0, \frac{1}{m}]$. Therefore $x \notin \bigcap_{m \in z^+} [0, \frac{1}{m}]$. $\therefore$ if $x \in \bigcap_{m \in z^+} [0, \frac{1}{m}]$ then $x = 0$.

(b) $\bigcup\{(a,b) \mid (a,b) \subseteq [1,2]\} = (1, 2)$

proof:

Suppose $x \in (1, 2)$. Choose a and b so that $1 < a < x < b < 2$ Then $x \in (a, b)$. Therefore, $x \in \bigcup\{(a,b) \mid (a,b) \subseteq [1,2]\}$

Conversely, suppose $x \in$ Left. Side. Then there exist a & b such that $1 < a < b < 2$ such that $x \in (a, b)$. $\therefore$ $1 < x < 2$ so $x \in (1, 2)$.

4. (a) Prove that if R is a linear ordering relation then so is R^{-1}.

proof Suppose R is a linear ordering.

(a) Show R^{-1} is reflexive. Let $x \in D_{R^{-1}} = D_R$ Then $\langle x, x\rangle \in R$ because R is reflexive. Consequently, $\langle x, x\rangle \in R^{-1}$ so R^{-1} is reflexive

(b) Show R^{-1} is anti-symmetric. Suppose $\langle x, y\rangle, \langle y, x\rangle \in R^{-1}$ Then $\langle y, x\rangle, \langle x, y\rangle \in R$, Since R is anti-symmetric, $x = y$

(c) Show R^{-1} is transitive Suppose $\langle x, y\rangle, \langle y, z\rangle \in R^{-1}$ Then $\langle y, x\rangle, \langle z, y\rangle \in R$. Then since R is transitive $\langle z, x\rangle \in R$ which implies $\langle x, z\rangle \in R^{-1}$

(d) Show R^{-1} is connected. Suppose $x, y \in D_{R^{-1}} = D_R$ Then since R is connected, either $\langle x, y\rangle \in R$ or $\langle y, x\rangle \in R$ This implies, either $\langle y, x\rangle \in R^{-1}$ or $\langle x, y\rangle \in R^{-1}$, so R^{-1} is connected

(b) Give an example of a relation R which is a well-ordering relation but R^{-1} is not.

Let $R = \leq$ on $\mathbb{Z}^+$ Then $R^{-1} = \geq$

$\langle \mathbb{Z}^+, \leq\rangle$ is well-ordered but

$\langle \mathbb{Z}^+, \geq\rangle$ is not

5. Find two sets X and Y such that $X \approx Y$ but $(X - Y) \not\approx (Y - X)$.

let $X = Z^+$, $Y = Z^+ - \{0\}$

Then $X \approx Y$ by the mapping F

where $F(n) = n + 1$. But $X - Y = \{0\}$

and $Y - X = \emptyset$ so

$$X - Y \not\approx Y - X.$$