Solution

MA 387 Exam 1 February 26, 1982

Name ______________________

(20) 1. Prove that for all sets A, B and C,

$$C \subseteq A \cap B \text{ iff } (C \cap \bar{A}) \cup (C \cap \bar{B}) = \phi.$$

proof Suppose $C \subseteq A \cap B$ (Method of proof: we shall show that if $x \in (C \cap \bar{A}) \cup (C \cap \bar{B})$ then we get a contradiction.)

Suppose $x \in (C \cap \bar{A}) \cup (C \cap \bar{B})$. Then, by the definition of $\cup$, $x \in C \cap \bar{A}$ or $x \in C \cap \bar{B}$. If $x \in C \cap \bar{A}$, then by the definition of $\cap$ and complement $x \in C$ and $x \notin A$. But this contradicts $C \subseteq A \cap B$. Similarly, if $x \in C \cap \bar{B}$ then $x \in C$ and $x \notin B$, which again contradicts $C \subseteq A \cap B$.

Conversely, suppose $(C \cap \bar{A}) \cup (C \cap \bar{B}) = \emptyset$ and $x \in C$ (show $x \in A \cap B$. We shall show we get a contradiction if $x \notin A \cap B$) Suppose $x \notin A \cap B$. Then by the definition of complement, $x \in \overline{A \cap B} = \bar{A} \cup \bar{B}$. Therefore, either $x \in \bar{A}$ or $x \in \bar{B}$. By hypotheses, $x \in C$ so if $x \in \bar{A}$, then $x \in C \cap \bar{A} \subseteq (C \cap \bar{A}) \cup (C \cap \bar{B}) = \emptyset$ which is a contradiction. Similarly, if $x \in \bar{B}$, then $x \in C \cap \bar{B} \subseteq (C \cap \bar{A}) \cup (C \cap \bar{B}) = \emptyset$. Again a contradiction. So $x \in A \cap B$

$\therefore C \subseteq A \cap B$.

(10) 2. Show that for all sets A and B, $A^B \subseteq \mathcal{P}(B \times A)$.

$A^B = \{f \mid f : B \to A\}$. Thus, f is a set of ordered pairs $\langle b, a \rangle$ such that $b \in B$ + $a \in A$.

$\mathcal{P}(B \times A)$ = set of <u>all</u> ordered pairs $\langle b, a \rangle$ such that $b \in B$ and $a \in A$.

$\therefore \quad A^B \subseteq \mathcal{P}(B \times A)$

(15) 3. Define a relation ρ on $\underline{Z}$ such that for all a, b ϵ $\underline{Z}$

a ρ b iff a - b is divisible by 5.

Show that ρ is an equivalence relation and describe the equivalence classes.

1. Show ρ is reflexive. Suppose $a \rho b$ then $a - b$ is divisible by 5 $\therefore$ $b - a$ is also divisible by 5. so $b \rho a$

2. Show ρ is transitive. Suppose $a \rho b$ & $b \rho c$ then $a - b$ is divisible by 5 and $b - c$ is divisible by 5. so $(a - b) + (b - c) = a - c$ is divisible by 5 $\therefore$ $a \rho c$.

3. There are 5 equivalence classes

$[0] = \{x \mid x \text{ is divisible by } 5\}$

$[1] = \{x \mid x - 1 \text{ is divisible by } 5\}$

$[2] = \{x \mid x - 2 \text{ is divisible by } 5\}$

$[3] = \{x \mid x - 3 \text{ is divisible by } 5\}$

$[4] = \{x \mid x - 4 \text{ is divisible by } 5\}$

(35) 4. Suppose $f : \underline{R} \to \underline{R}$ such that for each $x \in \underline{R}$, $f(x) = 3 + 2x$.

(a) Show f is a bijection (1-1 and onto).

1. f is 1-1 Suppose $f(x_1) = f(x_2)$ then $3 + 2x_1 = 3 + 2x_2$

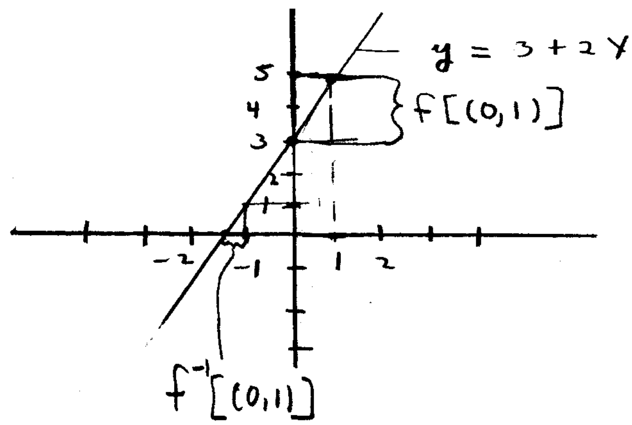
(12) $\longrightarrow 2x_1 = 2x_2 \longrightarrow x_1 = x_2$

2. f is onto. Given $y \in \underline{R}$, find $x \in \underline{R}$ so that $y = 3 + 2x$. Take $x = \frac{1}{2}(y-3)$.

(b) Let $(0,1) = \{x \in \underline{R} \mid 0 < x < 1\}$. Calculate $f[(0,1)]$ and $f^{-1}[(0,1)]$.

(12) $f[(0,1)] = (3,5)$

$f^{-1}[(0,1)] = (-\frac{3}{2}, -1)$



(c) If $g : \underline{R} \to \underline{R}$ such that for each $x \in \underline{R}$, $g(x) = x^2$, calculate $f \circ g(2)$ and $g \circ f(2)$.

$f \circ g(2) = f(2^2) = 3 + 2 \cdot 4 = 11$

(11) $g \circ f(2) = g(7) = 49$

$$\left[\begin{array}{l} f \circ g(x) = f(g(x)) = 3 + 2x^2 \\ g \circ f(x) = g(f(x)) = (3+2x)^2 \end{array}\right]$$

(20) 5. If $f \subseteq A \times B$, prove that f is a function iff $f \circ f^{-1} \subseteq i_B$.

proof:

Suppose f is a function and $\langle x, y \rangle \in f \circ f^{-1}$ (show $x = y$). Then by the definition of $\circ$, there is a z such that $\langle x, z \rangle \in f^{-1}$ & $\langle z, y \rangle \in f$. By the definition of f^{-1}, $\langle x, z \rangle \in f^{-1} \rightarrow \langle z, x \rangle \in f$. Since f is a function, $\langle z, x \rangle, \langle z, y \rangle \in f \rightarrow x = y$

Conversely, suppose $f \circ f^{-1} \subseteq i_B$ and $\langle x, y \rangle, \langle x, z \rangle \in f$. (show $y = z$). By the definition of f^{-1}, $\langle x, y \rangle \in f \rightarrow \langle y, x \rangle \in f^{-1}$. $\langle y, x \rangle \in f^{-1}$ and $\langle x, z \rangle \in f$ implies $\langle y, z \rangle \in f \circ f^{-1}$. By hypotheses, $f \circ f^{-1} \subseteq i_B$. $\therefore \langle y, z \rangle \in i_B$ which implies $y = z$.