Solution

MA 387 Exam 2 October 4, 1996

Name________________________ Stud. No.________________

1. (15) Show that the statement, $\bigcap \mathcal{P}(X) = X$ for all X, is **NOT** a theorem of set theory.

Let $X = \{1\}$ then $\mathcal{P}(X) = \{\emptyset, \{1\}\}$

and $\bigcap \mathcal{P}(X) = \emptyset \neq X$

2. (15) Suppose that X is a set of functions that is linearly ordered by the subset relation, $\subseteq$. Prove that $\bigcup X$ is a function.

Proof: If $u \in \bigcup X$ then there is a function $f \in X$ such that $u \in f$. Therefore u is an ordered pair. To prove $\bigcup X$ is a function, must show that if $\langle u, v\rangle, \langle u, w\rangle \in \bigcup X$ then $v = w$. Suppose $\langle u, v\rangle, \langle u, w\rangle \in \bigcup X$, then there is an $f_1, f_2 \in X$ such that $\langle u, v\rangle \in f_1$ and $\langle u, w\rangle \in f_2$. Since X is linearly ordered either $f_1 \subseteq f_2$ or $f_2 \subseteq f_1$. If $f_1 \subseteq f_2$ then $\langle u, v\rangle, \langle u, w\rangle \in f_2$. Since f_2 is a function $v = w$. Similarly if $f_2 \subseteq f_1$.

3. (15) Define a relation ρ on $\mathbb{Z}^+$ so that for all $x, y \in \mathbb{Z}^+$, $x\rho y$ iff $1\rho 1$ or x and y have a common factor which is greater than 1. Prove that ρ is **NOT** a partial ordering relation on $\mathbb{Z}^+$.

$2 \rho 4$ and $4 \rho 2$ but $2 \neq 4$ $\therefore$

ρ is not antisymmetric.

$2 \rho 6$ and $6 \rho 9$ but NOT $2 \rho 9$

$\therefore$ ρ is not transitive.

4. (15) Prove using mathematical induction that $2^{2n} - 1$ is divisible by 3, for all $n \geq 0$.

Base step: $k = 0$, $2^0 - 1 = 0$ which is divisible by 3

Inductive step: Suppose IH $2^{2m} - 1$ is divisible by 3, $m \geq 0$

(Prove $2^{2(m+1)} - 1$ is divisible by 3.

$$2^{2(m+1)} - 1 = 2^{2m+2} - 1 \quad \text{algebra}$$

$$= 4 \cdot 2^{2m} - 1 \quad \text{algebra}$$

$$= 4(3k+1) - 1 \quad (2^{2m} - 1 = 3k \text{ IH})$$

$$= 12k + 4 - 1 \quad \text{alg}$$

$$= 12k + 3 \quad \text{alg}$$

$$= 3(4k+1) \quad \text{alg.}$$

$\therefore$ $2^{2(m+1)} - 1$ is divisible by 3

so thm holds by M.I.

5. (20) Define a relation R on $\mathbb{Z}^+ \times \mathbb{Z}^+$ so that for all $< a, b >, < c, d > \in \mathbb{Z}^+ \times \mathbb{Z}^+$,

$$< a, b > R < c, d > \text{ iff } b < d, \text{ or } b = d \text{ and } a \leq c.$$

Prove that R is a linear ordering on $\mathbb{Z}^+ \times \mathbb{Z}^+$.

(1) R is reflexive: $a = a$ & $b = b$ $\therefore \langle a, b\rangle R \langle a, b\rangle$.

(2) R is antisymmetric: suppose (a) $\langle a, b\rangle R \langle c, d\rangle$ and (b) $\langle c, d\rangle R \langle a, b\rangle$

by (a) $a < c$ or $a = c$ and $b \leq d$

by (b) $c < a$ or $c = a$ and $d \leq b$.

If $a < c$ or $c < a$ (a) & (b) cannot both hold.

$\therefore$ $a = c$ and $b \leq d$ and $d \leq b$, the relation $\leq$ is ~~reflexive~~ antisymmetric $\therefore b = d$ so $\langle a, b\rangle = \langle c, d\rangle$.

(3) Transitive. Suppose (a) $\langle a, b\rangle R \langle c, d\rangle$ and $\langle c, d\rangle R \langle e, f\rangle$ then.

From (a) $a < c$ or $a = c$ & $b \leq d$

From (b) $c < e$ or $c = e$ & $d \leq f$.

If $a < c$, then by (b) either $c < e$ or $c = e$, in either case $a < e$ (by transitivity of $<$) so $\langle a, b\rangle R \langle e, f\rangle$.

Similarly if $c < e$ we get $\langle a, b\rangle R \langle e, f\rangle$.

Suppose $a = c$ & $b \leq d$, $c = e$ & $d \leq f$.

then we get $a = e$ and $b \leq f$ (Transitivity of $\leq$)

$\therefore$ $\langle a, b\rangle R \langle e, f\rangle$.

4 Connected: Suppose $\langle a, b\rangle$ & $\langle c, d\rangle \in \mathbb{Z}^+ \times \mathbb{Z}^+$

If $a < c$ then $\langle a, b\rangle R \langle c, d\rangle$ & if $c < a$, $\langle c, d\rangle R \langle a, b\rangle$

Suppose $a = e$ then either (i) $b \leq d$ or (ii) $d \leq b$

(i) $\Rightarrow \langle a, b\rangle R \langle c, d\rangle$ & (ii) $\Rightarrow \langle c, d\rangle R \langle a, b\rangle$

(b) l.u.b A = l.u.b $\{\langle n, 1\rangle : n \in \mathbb{Z}^+\} = \langle 1, 2\rangle$

6. (20) Suppose that f is a function that is defined as follows:

$$f(0) = 1,\ f(1) = 0,\ f(n+1) = 3f(n) - 2f(n-1),\ \text{for all } n \geq 1.$$

Prove using strong induction that $f(n) = 2 - 2^n$, for all $n \geq 0$.

Base Step: $k = 0$, $f(0) = 1$, $2 - 2^0 = 2 - 1 = 1$

$k = 1$, $f(1) = 0$, $2 - 2^1 = 2 - 2 = 0$.

Inductive Step suppose $f(p) = 2 - 2^p$, for all $p \leq m$

(Prove $f(m+1) = 2 - 2^{m+1}$).

$f(m+1) = 3f(m) - 2f(m-1)$ Def of f.

$= 3(2 - 2^m) - 2(2 - 2^{m-1})$ IH

$= 6 - 3 \cdot 2^m - 4 + 2^m$ alg

$= 2 - 2 \cdot 2^m$ alg.

$= 2 - 2^{m+1}$ alg.

∴ The Thm follows by S. I.