Solution

Math 387 Exam II April 18, 1975

Name ______________________________

In what follows, N is the set of natural numbers, Q is the set of rational numbers, and R is the set of real numbers.

(20) 1. Prove by mathematical induction that for all $n \in \underline{N}$, $n \geq 1$,

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n} > \frac{2}{5}.$$

(i) If $n = 1$, the left side $= \frac{1}{1+1} = \frac{1}{2} > \frac{2}{5}$. So the statement is true

(ii) Suppose $\frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} > \frac{2}{5}$ (Induction Hypothes

Then, if n is replaced by $n+1$ in the left side we obtain:

$$\begin{aligned}
\text{Left Side} &= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} \\
&\geq \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{2n+2} + \frac{1}{2n+2} \\
&= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{2}{2n+2} \\
&= \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{n+1} \\
&> \frac{2}{5} \quad \text{by the induction hypothesis}
\end{aligned}$$

Therefore, it follows by mathematical induction that

$$\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} > \frac{2}{5}$$

for all $n \in \underline{N}$, $n \geq 1$.

2. For each $n, m \in \underline{N}$, $m \neq 0$, let $rm(n,m)$ be the remainder when n is divided by m. Define a relation R on $\underline{N}$ as follows: if $n, m \in \underline{N}$

$$nRm \text{ iff } rm(n,3) < rm(m,3), \text{ or}$$

$$rm(n,3) = rm(m,3) \text{ and } n \leq m.$$

(10) (i) Give an inductive definition of $rm(n,m)$.

$$rm(0,m) = 0$$

$$rm(n+1,m) = \begin{cases} rm(n,m)+1 & \text{if } \frac{n+1}{m} \notin \underline{N} \\ 0 & \text{if } \frac{n+1}{m} \in \underline{N} \end{cases}$$

(10) (ii) List the elements of $\underline{N}$ as ordered by R.

$$\langle 0, 3, 6, 9, 12, \ldots, 1, 4, 7, 10, 13, \ldots, 2, 5, 8, 11, 14, \ldots \rangle$$

(10) (iii) Give the order type of $\langle \underline{N}, \underline{R} \rangle$.

$$\overline{\langle \underline{N}, R \rangle} = \omega + \omega + \omega$$

$$= \omega \cdot 3$$

(20) 2 (iv) Prove that R well orders $\underline{N}$.

(1) R is reflexive because $rm(n,3) = rm(n,3)$ for all $n \in \underline{N}$ so $n R n$.

(2) R is antisymetric. Suppose $n, m \in \underline{N}$ and $n R m$ and $m R n$.

Case 1 $rm(n,3) < rm(m,3)$ & $rm(m,3) < rm(n,3)$
This is impossible

Case 2 $rm(n,3) < rm(m,3)$, $rm(m,3) = rm(n,3)$, & $m \leq n$. (Similar if n & m are interchanged.) Again this cannot happen.

Case 3 $rm(n,3) = rm(m,3)$, $n \leq m$, and $m \leq n$. Then since, $\leq$ is antisymmetric, $n = m$.

(3) R is transitive. Suppose $n, m, p \in \underline{N}$ and $n R m$ & $m R p$

Case 1 $rm(n,3) < rm(m,3)$ & $rm(m,3) < rm(p,3)$. Then since $<$ is transitive, $rm(n,3) < rm(p,3)$ so $n R p$.

Case 2 $rm(n,3) < rm(m,3)$ & $rm(m,3) = rm(p,3)$, $m \leq p$. Then $rm(n,3) < rm(p,3)$ so $n R p$. Case is similar if $rm(n,3) = rm(m,3)$, $n \leq m$, & $rm(m,3) < rm(p,3)$

Case 3 $rm(n,3) = rm(m,3)$, $n \leq m$, $rm(m,3) = rm(p,3)$ $m \leq p$. Then $rm(n,3) = rm(p,3)$ and since $\leq$ is transitive, $n \leq p$. So $n R p$

(4) Let X be a non-empty subset of $\underline{N}$. First let s be the smallest element in $\{0,1,2\}$ such that there is an $n \in X$ with $rm(n,3) = s$. Let $Y = \{n \in X : rm(n,3) = s\}$. By Definition of R, for each $n \in Y$ + $m \in X - Y$, $n R m$. The R-smallest element of X = R-smallest element of Y = $\leq$-smallest element of Y.

(30) 3. Each of the following sets have cardinal number $\aleph_0$, c, or f. State which it is.

(i) The set of all points in 3-dimensional space. (Ie. $\{\langle u, v, w\rangle \mid u, v, w \in \underline{R}\}$.)

Cardinal Number = c because $c \cdot c \cdot c = c$.

(ii) The set of all $u \in \underline{R}$ which have the property that $u^n \in \underline{Q}$ for some $n \in \underline{N}$, $n \geq 1$.

Cardinal Number = $\aleph_0$ because it is an infinite subset of the set of algebraic numbers

(iii) The set of all functions which map the unit interval (0,1) onto itself. ($(0,1) = \{u \in \underline{R} \mid 0 < u < 1\}$.)

Cardinal Number = f because it contains the set of all 1-1 functions mapping (0,1) onto (0,1), which has cardinal number f, and it is a subset of $(0,1)^{(0,1)}$, which also has cardinal number f.

iv) The set of all constant real valued functions.

(Ie. $\{g \mid (\exists u \in \underline{R})\, g : \underline{R} \to \{u\}\}$.)

Cardinal Number = c because there is a 1-1 function mapping the given set onto $\underline{R}$.

v) The set of all functions from $\underline{R}$ into $\underline{Q}$.

Cardinal Number = f because $\overline{\overline{\underline{Q}^{\underline{R}}}} = \aleph_0^{c} = f$

vi) The set of all functions from $\underline{Q}$ into $\underline{R}$.

Cardinal Number = c because $\overline{\overline{\underline{R}^{\underline{Q}}}} = c^{\aleph_0} = c$