MA 387 Solution Exam 2 April 16, 1982

Name ____________________

(10) 1. Calculate each of the following:

a) $\cap\{(a,b) \mid a < 0 \text{ and } b \geq 2\}$ = $[0, 2)$

b) $\cup\{(a,b) \mid [a,b] \subseteq (1,2)\}$ = $(1, 2)$

(15) 2. Prove that $f[\bigcap_{i\in I} A_i] \subseteq \bigcap_{i\in I} f[A_i]$

$x \in f[\bigcap_{i\in I} A_i]$ iff $(\exists y)(y \in \bigcap_{i\in I} A_i \wedge x = f(y))$

iff $(\exists y)(\forall i \in I)[y \in A_i \wedge x = f(y)]$

$\rightarrow (\forall i \in I)(\exists y)[y \in A_i \wedge x = f(y)]$

iff $(\forall i \in I)\ x \in f[A_i]$

iff $x \in \bigcap_{i\in I} f[A_i]$

(25) 3. Define a relation ρ such that for all sets A and B,

$A \rho B$ iff $A = A \cup B$

a) Prove that ρ is a partial ordering relation.

1. reflexive: $A \rho A$ because $A = A \cup A$.

2. anti-symmetric:

Suppose $A \rho B$ and $B \rho A$. Then

$A = A \cup B$ and $B = B \cup A$. $A \cup B = B \cup A \rightarrow A = B$

3. transitive:

Suppose $A \rho B$ and $B \rho C$ Then

$A = A \cup B$ and $B = B \cup C$. Therefore

$$\begin{aligned} A &= A \cup B \\ &= A \cup (B \cup C) \\ &= (A \cup B) \cup C \\ &= A \cup C \end{aligned}$$

$\therefore A \rho C$.

b) Give an example of an infinite set x such that <x,ρ> is a well ordered set, where ρ is defined above.

Let $X = \{ (0, \frac{1}{n}) \mid n \in \underline{N}^+ \}$

Note: $A \rho B$ iff $A \supseteq B$

$X = \langle (0,1), (0, \frac{1}{2}), (0, \frac{1}{3}), (0, \frac{1}{4}), (0, \frac{1}{5}) \cdots \rangle$

(25) 4. Consider the sequence

$$x_1 = \sqrt{2},\ x_2 = \sqrt{2 + \sqrt{2}},\ x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}},\ \ldots$$

a) Give an inductive definition of x_n.

$$x_1 = \sqrt{2}$$

$$x_{n+1} = \sqrt{2 + x_n}$$

b) Prove that $x_n < 2$ for all $n \geq 1$.

If $n = 1$, $x_1 = \sqrt{2} < 2$.

IH. Suppose $x_n < 2$ then

$$x_{n+1} = \sqrt{2 + x_n} < \sqrt{2 + 2} = 2$$

So it follows by M. I. that $x_n < 2$ for all $n \geq 1$

(10) 5. Prove that for all sets A, if $\underline{N} \precsim A$ and $x \notin A$ then $A \cup \{x\} \approx A$.

Clearly (1) $A \precsim A \cup \{x\}$. Since $\underline{N} \precsim A$, there is a subset $B \subseteq A$ such that $f: \underline{N} \approx B$. Define $g: A \cup \{x\} \longrightarrow A$ as follows: $g(x) = b_0$, $\forall n \in \omega$, $g(b_n) = b_{n+1}$. If $u \in A - B$, $g(u) = u$. Then g is a 1-1 function so (2) $A \cup \{x\} \precsim A$. It follows from (1) and (2) that $A \cup \{x\} \approx A$.

(15) 6. Prove using mathematical induction that

$$(1 - x)(1 + x)(1 + x^2)(1 + x^4)\cdots(1 + x^{2^n}) = (1 - x^{2^{n+1}})$$

for all $n \geq 0$.

$n = 0$ $\qquad (1-x)(1+x) = 1 - x^2 = 1 - x^{2^{(0+1)}}$

I.H. Suppose $\qquad (1-x)(1+x)(1+x^2)\cdots(1+x^{2^n}) = (1-x^{2^{n+1}})$

Then

$$(1-x)(1+x)(1+x^2)\cdots(1+x^{2^n})(1+x^{2^{n+1}}) =$$

$$= (1-x^{2^{n+1}})(1+x^{2^{n+1}}) \quad \text{by I.H.}$$

$$= (1 - x^{2^{n+2}}) \quad \text{factoring}$$