MA 387 Solution Final Exam December 9, 1996

Name ____________________ Stud. No. ____________________

1. (20) If $A = \{a, b, c\}$ and $B = \{0, 1\}$ calculate $A \times B$ and $\mathcal{P}(A)$.

$A \times B = \{\langle a,0\rangle, \langle a,1\rangle, \langle b,0\rangle, \langle b,1\rangle, \langle c,0\rangle, \langle c,1\rangle\}.$

$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

2. (20) Prove that for all sets A, B, and C, $(A \cup B) - C = (A - C) \cup (B - C)$.

Proof:

$x \in$ L.S iff $x \in A \cup B$ and $x \notin C$

iff $(x \in A$ or $x \in B)$ and $x \notin C$.

iff $(x \in A$ and $x \notin C)$ or $(x \in B$ and $x \notin C)$

iff $x \in A - C$ or $x \in B - C$.

3. (30). Let $n \in \mathbb{N}^+$ and let $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. Calculate each of the following.

(a) $\bigcup_{n \in \mathbb{N}^+} (0, 2 + \frac{1}{n})$. $= (0, 3)$

(b) $\bigcap_{n \in \mathbb{N}^+} (0, 2 + \frac{1}{n})$. $= (0, 2]$

4. (20) Prove that for all non-empty sets A, B, and C, if $A \sim B$, then $A^C \sim B^C$.

Proof Suppose $g : A \sim B$. Define F on A^C such that for $h \in A^C$, $F(h) = k$ where for each $x \in C$, $k(x) = g(h(x))$.

F is 1-1 Suppose $F(h_1) = F(h_2) = k$. Then for each $x \in C$, $k(x) = g(h_1(x)) = g(h_2(x))$. Since g is 1-1, $h_1(x) = h_2(x)$ $\therefore h_1 = h_2$.

F is onto: Suppose $k \in B^C$. Define $h \in A^C$ so that for each $x \in C$. $h(x) = g^{-1}(k(x))$ (g is 1-1 & onto). Then $F(h) = k$ because $g(h(x)) = g(g^{-1}(k(x))) = k(x)$.

5 (35). Define a relation ρ on $\mathbb{R} \times \mathbb{R}$ so that for all $< a, b >, < c, d > \in \mathbb{R} \times \mathbb{R}$,

$$< a, b > \rho < c, d > \text{ iff } a \leq c \text{ and } b \leq d.$$

(a) Prove that ρ is a partial ordering on $\mathbb{R} \times \mathbb{R}$, but not a linear ordering.

ρ is reflexive because $a \leq a$ & $b \leq b$ $\therefore \langle a,b \rangle \rho \langle a,b \rangle$.
($\leq$ is reflexive).
ρ is anti-symmetric because if $\langle a,b \rangle \rho \langle c,d \rangle$ and $\langle c,d \rangle \rho \langle a,b \rangle$ then $a \leq c$ $b \leq d$, $c \leq a$ & $d \leq b$. But $\leq$ is ante-symmetric $\therefore$ $a = c$ & $b = d$ so $\langle a,b \rangle = \langle c,d \rangle$.
ρ is transitive because if $\langle a,b \rangle \rho \langle c,d \rangle$ & $\langle c,d \rangle \rho \langle e,f \rangle$ then $a \leq c$, $b \leq d$, $c \leq e$ & $d \leq f$. but $\leq$ is transitive so $a \leq e$ & $b \leq f$ so $\langle a,b \rangle \rho \langle e,f \rangle$.
ρ is not a linear ordering because for example, we have that neither $\langle 1,3 \rangle \rho \langle 2,2 \rangle$ nor $\langle 2,2 \rangle \rho \langle 1,3 \rangle$.

(b) For each $m \in \mathbb{R}^+$, let $X_m = \{< a, ma >: a \in \mathbb{R}\}$, then $X_m \subseteq \mathbb{R} \times \mathbb{R}$. Prove that for each $m \in \mathbb{R}^+$, ρ is a linear ordering on X_m.

sufficient to show that ρ is connected on X_m.

Suppose $\langle a, ma \rangle$, $\langle b, ma \rangle \in X_m$ since $\leq$ is connected on $\mathbb{R}$, either $a \leq b$ or $b \leq a$.
If $a \leq b$ then $ma \leq mb$ so $\langle a, ma \rangle \rho \langle b, ma \rangle$
If $b \leq a$ then $mb \leq ma$ so $\langle b, ma \rangle \rho \langle a, ma \rangle$
$\therefore$ ρ is connected on X_m.

6. (15) Prove using mathematical induction that $2^n > n^2 + 19$, for all $n \geq 6$.

Base step: $n = 6$. L.S $= 2^6 = 64$; R.C $= 36 + 19 = 55$ ✓

Induction step suppose I.H $2^m > m^2 + 19$, $m \geq 6$.

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\begin{aligned}
2^{m+1} &= 2 \cdot 2^m && \text{(prop of exponents)} \\
&> 2(m^2 + 19) && \text{(IH)}. \\
&= 2m^2 + 38 && \text{(alg)}. \\
&\geq m^2 + 2m + 1 + 37 && (m \geq 6 \therefore m^2 > 2m. \\
&> (m+1)^2 + 19 && \text{(alg)}.
\end{aligned}
$$

Thm follows by M. I.

7. (15) Suppose a Function f is defined so that $f(0) = 1$ and $f(n+1) = \sum_{i=0}^{n} f(i)$. Prove using strong induction that $f(n) = 2^{n-1}$, for $n \geq 1$. (Hint: $\sum_{i=0}^{m-1} 2^i = 2^m - 1$).

Base step: $n = 1$. By Def, $f(1) = f(0) = 1$. & $2^{1-1} = 1$.

Induction step: suppose that $f(k) = 2^{k-1}$ for all k, $1 \leq k \leq m$.

$$
\begin{aligned}
f(m+1) &= \sum_{i=0}^{m} f(i) && \text{(Def.)}. \\
&= f(0) + f(1) + f(2) + \cdots f(m) && \text{(Def of } \Sigma. \\
&= 1 + 2^0 + 2^1 + 2^2 + \cdots + 2^{m-1} && (f(0) = 1 \ \text{IH}). \\
&= 1 + 2^m - 1 && \text{(sum above)} \\
&= 2^m.
\end{aligned}
$$

Thm follows by S. I.

8. (30) Prove each of the following. ($c = 2^{\aleph_0}$, $f = c^c$.)
(a) $c = \aleph_0^{\aleph_0} = c^{\aleph_0}$.

$$c \le \aleph_0^{\aleph_0} \le c^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0 \cdot \aleph_0} = 2^{\aleph_0} = c$$

Thm follows by the S-B. Thm.

(b) $f = f^{\aleph_0} = f^c$.

$$f \le f^{\aleph_0} \le f^c = (c^c)^c = c^{c \cdot c} = c^c = f.$$

Thm follows by the S-B Thm.

9. (15) Find the cardinal number of the set of all functions $g : \mathbb{R} \to \mathbb{R}$ such that g is differentiable. (Hint: Every differentiable function is continuous.)

Let X = set of differentiable function

The cardinal No of the constant functions is c

+ every constant function is differentiable $\therefore c \le \overline{\overline{X}}$.

Also, every differentiable function is continuous and we have proved that the cardinality of the set of continuous functions is c $\therefore \overline{\overline{X}} \le c$.

It follows from the S-B thm that $\overline{\overline{X}} = c$.

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