Math 387 Final Exam May 1975

Name ______________________________

(25) 1. Show that for all sets A and B,

$$\overline{(A \cup B) - (A \cap B)} = (A \cup \bar{B}) - (A \cap \bar{B}).$$

$$\text{L.S.} = \overline{(A \cup B) - (A \cap B)}$$

$$= \overline{(A \cup B) \cap \overline{(A \cap B)}}$$

$$= \overline{(A \cup B) \cap (\bar{A} \cup \bar{B})}$$

$$= \overline{(A \cup B)} \cup \overline{(\bar{A} \cup \bar{B})}.$$

$$= (\bar{A} \cap \bar{B}) \cup (A \cap B).$$

$$= (\bar{A} \cup A) \cap (\bar{A} \cup B) \cap (\bar{B} \cup A) \cap (\bar{B} \cup B)$$

$$= (\bar{A} \cup B) \cap (\bar{B} \cup A)$$

$$= (A \cup \bar{B}) \cap (\bar{A} \cup B)$$

$$= (A \cup \bar{B}) \cap \overline{(A \cap \bar{B})}$$

$$= (A \cup \bar{B}) - (A \cap \bar{B}).$$

(15) 2. (a) Show that for all sets A and B, $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$

If $x \in \mathcal{P}(A) \cup \mathcal{P}(B)$ then $x \in \mathcal{P}(A)$ or $x \in \mathcal{P}(B)$

If $x \in \mathcal{P}(A)$ then $x \subseteq A \subseteq A \cup B$ so $x \in \mathcal{P}(A \cup B)$

If $x \in \mathcal{P}(B)$ then $x \subseteq B \subseteq A \cup B$ so $x \in \mathcal{P}(A \cup B)$

Thus, in either case $x \in \mathcal{P}(A \cup B)$.

So $\mathcal{P}(A) \cup \mathcal{P}(B) \subseteq \mathcal{P}(A \cup B)$.

(15) (b) Show by constructing a counter example that it is not the case that $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$ for all sets A and B.

Suppose $A = \{1\}$ and $B = \{2\}$ then

$\mathcal{P}(A) = \{\phi, \{1\}\}$, $\mathcal{P}(B) = \{\phi, \{2\}\}$ so

$\mathcal{P}(A) \cup \mathcal{P}(B) = \{\phi, \{1\}, \{2\}\}$.

But $A \cup B = \{1, 2\}$

$\mathcal{P}(A \cup B) = \{\phi, \{1\}, \{2\}, \{1,2\}\}$.

$\{1,2\} \in \mathcal{P}(A \cup B)$ but $\{1,2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

$\therefore \mathcal{P}(A \cup B) \not\subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

(20) 3. If $A = \{(a,b) \mid (0,1) \subseteq (a,b)\}$, calculate $\bigcap A$ and $\bigcup A$.

Assuming a and b are real numbers.

$$\bigcup A = \{x : x \in (a,b) \text{ for some } a,b \in \mathbb{R},\ a<b, \text{ and } (0,1) \subseteq (a,b)\}.$$

$$= \mathbb{R}.$$

$$\bigcap A = \{x : x \in (a,b) \text{ for all } a,b \in \mathbb{R},\ a<b \text{ and } (0,1) \subseteq (a,b)\}$$

$$= (0,1).$$

(10) 4. Calculate the cardinality of the set of all circles in the plane with centers on the x-axis. Give reasons for your answer.

Center of the circle is of the form $(x, 0)$ where $x \in \mathbb{R}$ and the radius could be any $r \in \mathbb{R}^+$.

$\therefore$ cardinality is $\overline{\overline{\mathbb{R}}} \cdot \overline{\overline{\mathbb{R}^+}} = c^2 = c$.

5. Suppose g is a real valued function such that for each $x \in \mathbb{R}$,

$$g(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ x + 1 & \text{if } x < 0. \end{cases}$$

(30) (a) Show that g is a bijection.

Show g is 1-1, ie. if $g(x) = g(y)$ then $x = y$.

Case 1 $x, y \geq 0$ then $g(x) = 2x+1$ & $g(y) = 2y+1$

so if $g(x) = g(y)$, $2x+1 = 2y+1 \Longrightarrow x = y$.

Case 2 $x, y < 0$ then $g(x) = x+1$ & $g(y) = y+1$

so if $g(x) = g(y)$ $x+1 = y+1 \Longrightarrow x = y$.

Case 3 $x \geq 0, y < 0$ then $g(x) = 2x+1$ $g(y) = \cancel{y}+1$

then $g(x) \neq g(y)$ because $g(x) \geq 1$ & $g(y) < 1$

Case 4 $x < 0$ & $y \geq 0$ similarly to Case 3, this cannot happen

$\therefore$ g is 1-1.

(10) (b) Calculate $g[(-1,1)]$ and $g^{-1}[(-1,1)]$.

$g[(-1,1)] = (0, 3).$

$g^{-1}[(-1,1)] = (-2, 0).$

(10) (c) If h is a real valued function such that $h(x) = x^2$ for each $x \in \mathbb{R}$, calculate $g \circ h(-1)$ and $h \circ g(-1)$.

$g \circ h(-1) = g(h(-1)) = g(1) = 3.$

$h \circ g(-1) = h(g(-1)) = h(0) = 1$

6. Consider the sequence:

$x_1 = \sqrt{2}$, $x_2 = \sqrt{2 + \sqrt{2}}$, $x_3 = \sqrt{2 + \sqrt{2 + \sqrt{2}}}$, etc.

(15) (a) Give an inductive definition of x_n for $n \geq 1$.

$$x_1 = \sqrt{2}, \quad x_{n+1} = \sqrt{2 + x_n}$$

(20) (b) Prove that $x_n < 2$ for all $n \geq 1$. ($\sqrt{2} = 1.414\ldots$.)

$x_1 = \sqrt{2} < 2$.

Suppose (*) $x_n < 2$.

Then $x_{n+1} = \sqrt{2 + x_n} < \sqrt{2+2}$ by (*)

$= \sqrt{4}$

$= 2$.

$\therefore x_{n+1} < 2$.

It follows by math induction that $x_n < 2$ for all $n \geq 1$.

(30) 7. State in your own words the characteristic properties of the following sets.

(a) $\langle \underline{N}, \leq \rangle$

$\emptyset \in \mathbb{N}$ and every other element of $\mathbb{N}$ is of the form $m \cup \{m\}$ where $m \in \mathbb{N}$. and $m < m \cup \{m\}$.

(b) $\langle \underline{Q}, \leq \rangle$

Every element of $\mathbb{Q}$ is a ratio of integers of the form $\frac{m}{n}$ where $n \neq 0$.

+ $\frac{m_1}{n_1} \leq \frac{m_2}{n_2}$ iff $m_1 n_2 \leq m_2 n_1$.

(c) $\langle \mathbb{R}, \leq \rangle$

Each element of $\mathbb{R}$ is a Cauchy sequence of rational numbers. +.

$r_1 \leq r_2$ iff the limit of the Cauchy sequence representing r_1 is $\leq$ the limit of the Cauchy sequence representing r_2