MA 387 Final Exam May 6, 1982

Name ______________________________

(15) 1. Show that for all sets X, Y and Z
$(X \cup Y \cup Z) - (Y \cap Z) = (X \cap \overline{Y}) \cup (X \cap \overline{Z}) \cup (Y \cap \overline{Z}) \cup (Z \cap \overline{Y})$

$$\begin{aligned}
L.S. &= (X \cup Y \cup Z) \cap \overline{(Y \cap Z)} \\
&= (X \cup Y \cup Z) \cap (\overline{Y} \cup \overline{Z}) \\
&= (X \cap \overline{Y}) \cup (Y \cap \overline{Y}) \cup (Z \cap \overline{Y}) \cup (X \cap \overline{Z}) \cup (Y \cap \overline{Z}) \cup (Z \cap \\
&= (X \cap \overline{Y}) \cup (Z \cap \overline{Y}) \cup (X \cap \overline{Z}) \cup (Y \cap \overline{Z}) \\
&= (X \cap \overline{Y}) \cup (X \cap \overline{Z}) \cup (Y \cap \overline{Z}) \cup (Z \cap \overline{Y})
\end{aligned}$$

(15) 2. Show that for all sets X, Y and Z
$(X \cap Y) \cup Z = X \cap (Y \cup Z)$ iff $Z \subseteq X$.

$\rightarrow$ Suppose $u \in Z$ then $u \in (X \cap Y) \cup Z = X \cap (Y \cup Z)$ which implies that $u \in X$ from the definition of $\cap$.
$\therefore u \in X$ so $Z \subseteq X$

$\leftarrow$ suppose $Z \subseteq X$. then

$$\begin{aligned}
(X \cap Y) \cup Z &= (X \cup Z) \cap (Y \cup Z) \quad \text{(distributive Law)} \\
&= X \cap (Y \cup Z) \quad (\text{since } Z \subseteq X,\ X \cup Z = X)
\end{aligned}$$

(10) 3. If $X = \{a, \{a\}, \{\{a\}\}\}$, calculate $\mathcal{P}(X)$.

$$\mathcal{P}(X) = \{\emptyset, \{a\}, \{\{a\}\}, \{\{\{a\}\}\}, \{a, \{a\}\}, \{a, \{\{a\}\}\}, \{\{a\}, \{\{a\}\}\}, X\}.$$

(15) 4. If R, S, and T are relations show that
$R \circ (S \cup T) = (R \circ S) \cup (R \circ T)$.

$$\begin{aligned}
\langle u,v\rangle \in R\circ(S\cup T) &\text{ iff } (\exists w)(\langle u,w\rangle \in R \ \&\ \langle w,v\rangle \in S\cup T)\\
&\text{ iff } (\exists w)[\langle u,w\rangle \in R \ \&\ (\langle w,v\rangle \in S \text{ or } \langle w,v\rangle \in T)]\\
&\text{ iff } (\exists w)[(\langle u,w\rangle \in R \ \&\ \langle w,v\rangle \in S)\\
&\qquad \text{ or } (\langle u,w\rangle \in R \ \&\ \langle w,v\rangle \in T)]\\
&\text{ iff } (\exists w)(\langle u,w\rangle \in R \ \&\ \langle w,v\rangle \in S)\\
&\qquad \text{ or } (\exists w)(\langle u,w\rangle \in R \ \&\ \langle w,v\rangle \in T)\\
&\text{ iff } \langle u,v\rangle \in R\circ S \text{ or } \langle u,v\rangle \in R\circ T\\
&\text{ iff } \langle u,v\rangle \in (R\circ S)\cup(R\circ T).
\end{aligned}$$

$$\therefore\ R\circ(S\cup T) = (R\circ S)\cup(R\circ T).$$

(25) 5. Four relations, ρ_1, ρ_2, ρ_3 and ρ_4 are defined below. The domain of each of ρ_1, ρ_2 and ρ_3 is $\underline{R}$ and the domain of ρ_4 is $[0,1)$.

$u\rho_1 v$ iff $u < v < u + 1$

$u\rho_2 v$ iff $u^2 + v^2 = 1$

$u\rho_3 v$ iff $v = u^2$

$u\rho_4 v$ iff $v = \frac{u}{1 - u}$

(a) Which of the above relations are functions? Give reasons for your answer.

ρ_1 is not a function because there is more than one real number v so that $u < v < u+1$.

ρ_2 is not a function because if $u \in [0,1)$ then $v = \pm\sqrt{1-u^2}$ satisfies $u \rho_2 v$.

ρ_3 and ρ_4 are functions because for each $u \in [0,1)$ there is exactly one v satisfying the given equations.

(b) Which of the functions are 1-1 functions? Give reasons for your answer.

ρ_3 is not 1-1 because if $u_1^2 = u_2^2$ then

$$u_1 = \pm u_2.$$

ρ_4 is 1-1 because if $\frac{u_1}{1-u_1} = \frac{u_2}{1-u_2}$ then

$$u_1(1-u_2) = u_2(1-u_1)$$

$$u_1 - u_1u_2 = u_2 - u_1u_2$$

$$\therefore \quad u_1 = u_2.$$

(15) 6. Prove that if F is a 1-1 function from A into B and if for all $u \subseteq A$, $G(u) = F[u]$, then G is a 1-1 function from $\mathcal{P}(A)$ into $\mathcal{P}(B)$.

Let $u, v \subseteq A$.

Suppose $G(u) = G(v)$, Then $F[u] = F[v]$.

If $u \neq v$, then either $(\exists w \in u)(w \notin v)$ or $(\exists w \in v)(w \notin u)$

If the first case holds, then $w \in u$ so $F(w) \in F[v]$ which implies that there is a $z \in v$ so that $F(w) = F(z)$. Since F is 1-1, $w = z$. This contradicts the assumption that $w \in u$ but $w \notin v$.

The proof of the second case is similar.

(15) 7. Define R on $\underline{N}$ so that for all $n, m \in \underline{N}$

$$nRm \quad \text{iff} \quad n < m + 2.$$

Show that R is reflexive, but R is not anti-symmetric nor transitive.

R is reflexive because for all $n \in \mathbb{N}$, $n < n + 2$.

R is not anti-symmetric. because.

$1 R 2$ because $1 < 2 + 2$.

$2 R 1$ because $2 < 1 + 2$

but $1 \neq 2$.

R is not transitive. because.

$3 R 2$ because $3 < 2 + 2$ and

$2 R 1$ because $2 < 1 + 2$ but

Not $3 R 1$ because $3 = 1 + 2$.

(20) 8. $\langle \underline{Q}, \underline{<} \rangle$ is a linearly ordered set. $\underline{Z}^+ \subseteq \underline{Q}$ and $\underline{Q}^+ \subseteq Q$.

(a) Does $\underline{Z}^+$ have a $\underline{<}$-first element? If so what is it?

yes, 1 is the $\leq$- first element of Z^+.

(b) Does $\underline{Q}^+$ have a $\underline{<}$-first element? If so what is it?

No, there is no smallest positive rational Number.

(c) Does $\underline{Z}^+$ have a $\underline{<}$-greatest lower bound? If so what is it?

yes, 0 is the $\leq$-greatest lower bound of Z^+

(d) Does $\underline{Q}^+$ have a $\underline{<}$-greatest lower bound? If so what is it?

yes, 0 is the $\leq$-greatest lower bound of Q^+

(10) 9. Calculate each of the following:

(a) $\bigcup_{n \in \underline{Z}^+} [1 + \frac{1}{n}, 3 - \frac{1}{n}] = [1, 3)$

(b) $\bigcap_{n \in \underline{Z}^+} (1 - \frac{1}{n}, 3 + \frac{1}{n}) = [1, 3]$

(15) 10. Prove that for all natural numbers $n \geq 6$,

$$2^n > n^2 + 15.$$

If $n = 6$, $2^n = 64$ and $n^2 + 15 = 6^2 + 15 = 51$ and $64 > 51$.

Suppose $2^n > n^2 + 15$, then

$$2^{n+1} = 2 \cdot 2^n > 2(n^2+15) = 2n^2 + 30$$
$$> n^2 + 2n + 30 \quad (n^2 > 2n$$
$$= (n+1)^2 + 29$$
$$> (n+1)^2 + 15.$$

Thus it follows by math induction that $2^n > n^2 + 15$ for all $n \geq 6$.

(15) 11. Prove that for all natural numbers $n \geq 1$,

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}.$$

If $n = 1$, L.S $= \frac{1}{2}$, R.S. $= 2 - \frac{3}{2} = \frac{1}{2}$.

Suppose $\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} = 2 - \frac{n+2}{2^n}$ (*)

Then

$$\frac{1}{2} + \frac{2}{2^2} + \frac{3}{2^3} + \cdots + \frac{n}{2^n} + \frac{n+1}{2^{n+1}} = 2 - \frac{n+2}{2^n} + \frac{n+1}{2^{n+1}} \quad \text{(by (*))}$$
$$= 2 + \frac{-2(n+2) + n+1}{2^{n+1}} \quad \text{alg}$$
$$= 2 + \frac{-2n - 4 + n + 1}{2^{n+1}}$$
$$= 2 + \frac{-n-3}{2^{n+1}}$$
$$= 2 - \frac{(n+1)+2}{2^{n+1}}$$

∴ the equation holds by Math Induction.

(30) 12. For each of the following sets, state whether its cardinal number is $\aleph_0$, c or f and give reasons for your answer.

(a) The set of all subsets of N.

$$\overline{\overline{\mathcal{P}(\mathbb{N})}} = 2^{\aleph_0} = c.$$

(b) The set of all finite subsets of N.

Count the set of all finite subsets of $\mathbb{N}$.

$\emptyset$ + I set of 1 element subsets $\{0\}, \{1\}, \{2\}, \cdots$;

II the set of two element subsets: $\{0,1\}, \{0,2\} \cdots, \{1,2\}, \{1,3\} \cdots$

III the set of three element subsets: $\{0,1,2\}, \{0,1,3\}, \cdots$

etc. $1 + \aleph_0 + \aleph_0 + \cdots = 1 + \aleph_0^2 = \aleph_0$.

(c) The set of all functions from Q to (0,1).

$$\overline{\overline{(0,1)^{\mathbb{Q}}}} = c^{\aleph_0} = c.$$

(d) The set of all functions from R to {0,1}.

$$\overline{\overline{\{0,1\}^{\mathbb{R}}}} = 2^{c} = f$$