NAME GRADING KEY

STUDENT ID ______

RECITATION INSTRUCTOR ______

RECITATION TIME ______

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DIRECTIONS

1. Write your name, student ID number, recitation instructor's name and recitation time in the space provided above. Also write your name at the top of pages 2, 3 and 4.
2. The test has four (4) pages, including this one.
3. Write your answers in the boxes provided.
4. You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
5. Credit for each problem is given in parentheses in the left hand margin.
6. No books, notes or calculators may be used on this exam.

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(16) 1. Find the derivatives of the following functions. (It is not necessary to simplify).

(a) $y = (1 + \cos^2 x)^6$.

* NPC but -1 pt if first answer is correct and there is an error in copying or simplifying

$$\frac{dy}{dx} = 6(1+\cos^2 x)^5 2\cos x(-\sin x)$$

$\boxed{-12\sin x\cos x(1+\cos^2 x)^5}$ 4

(b) $f(x) = \sin^{-1}(2x^2)$

$$f'(x) = \frac{1}{\sqrt{1-(2x^2)^2}}\,4x$$

$\boxed{\dfrac{4x}{\sqrt{1-4x^4}}}$ 4

(c) $H(x) = (1 + x^2)\tan^{-1} x$.

$$H'(x) = (1+x^2)\frac{1}{1+x^2} + 2x\tan^{-1}x$$

$\boxed{1 + 2x\tan^{-1}x}$ 4

(d) $y = \ln(\sin x^2)$

$$\frac{dy}{dx} = \frac{1}{\sin x^2}(\cos x^2)2x$$

$\boxed{\dfrac{2x\cos x^2}{\sin x^2}}$ 4

(6) 2. Find an equation of the tangent line to the curve $y^3 - x^2 = 4$ at the point $(2,2)$.

$y^3 - x^2 = 4$

$3y^2 \frac{dy}{dx} - 2x = 0$ ②

$\frac{dy}{dx} = \frac{2x}{3y^2}$

At $(x,y) = (2,2)$: $\frac{dy}{dx} = \frac{2\cdot 2}{3\cdot 2^2} = \frac{1}{3}$ ②

eq. of tan. line: $y - 2 = \frac{1}{3}(x-2)$

or

$y = \sqrt[3]{x^2+4}$

$\frac{dy}{dx} = \frac{1}{3}(x^2+4)^{-2/3} 2x$ ②

$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{3} 8^{-2/3} \cdot 2 \cdot 2 = \frac{1}{3}$ ②

② $\boxed{y - 2 = \frac{1}{3}(x-2)}$ $\boxed{6}$

(9) 3. If $x\sin y + \cos 2y = \cos y$, find $\frac{dy}{dx}$ by implicit differentiation.

$x \sin y + \cos 2y = \cos y$

$x \cos y \cdot \frac{dy}{dx} + \sin y$ ③ $+ (-\sin 2y)\cdot 2 \frac{dy}{dx}$ ② $= -\sin y \frac{dy}{dx}$ ②

$(x\cos y - 2\sin 2y + \sin y)\frac{dy}{dx} = -\sin y$

$\frac{dy}{dx} = \frac{-\sin y}{x\cos y - 2\sin 2y + \sin y}$

② $\boxed{\frac{dy}{dx} = -\frac{\sin y}{x\cos y - 2\sin 2y + \sin y}}$ $\boxed{9}$

(6) 4. Find the first and second derivatives of the function $h(x) = \sqrt{x^2+1}$.

$h(x) = \sqrt{x^2+1}$

$h'(x) = \frac{1}{2\sqrt{x^2+1}} 2x = \frac{x}{\sqrt{x^2+1}}$

$h''(x) = \frac{\sqrt{x^2+1}\cdot 1 - x\frac{1}{2\sqrt{x^2+1}} 2x}{x^2+1}$

$= \frac{x^2+1-x^2}{(x^2+1)^{3/2}}$ ← or ④ with rule * →

$\boxed{h'(x) = \frac{x}{\sqrt{x^2+1}}}$ ②

$\boxed{h''(x) = \frac{1}{(x^2+1)^{3/2}}}$ $\boxed{6}$

(8) 5. Find the derivative of the function $y = (\ln x)^x$.

$y = (\ln x)^x = e^{\ln(\ln x)^x} = e^{x\ln(\ln x)}$ ④

$\frac{dy}{dx} = e^{x\ln(\ln x)}\left[x \frac{1}{\ln x}\frac{1}{x} + \ln(\ln x)\right]$

$= (\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]$ ← or ④ with rule * →

$\boxed{(\ln x)^x\left[\frac{1}{\ln x} + \ln(\ln x)\right]}$ $\boxed{8}$

(9) 6. Find the exact value of each expression.

NPC

(a) $\sin^{-1}(-\frac{1}{2}) = y \iff \sin y = -\frac{1}{2}, -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$y = -\frac{\pi}{6}$

$\boxed{-\frac{\pi}{6}}$ ③

(b) $\tan^{-1} 1 = y \iff \tan y = 1, -\frac{\pi}{2} < y < \frac{\pi}{2}$

$y = \frac{\pi}{4}$

$\boxed{\frac{\pi}{4}}$ ③

(c) $\sin(\cos^{-1}\frac{4}{5}) = \sin y$ where $y = \cos^{-1}\frac{4}{5}$.

$y = \cos^{-1}\frac{4}{5} \iff \cos y = \frac{4}{5}, 0 \le y \le \pi$

$\sin y = +\sqrt{1-\cos^2 y} = \sqrt{1-(\frac{4}{5})^2} = \sqrt{\frac{9}{25}} = \frac{3}{5}$

(+$\sqrt{\ }$ because $\sin y \ge 0$ when $0 \le y \le \pi$)

$\boxed{\frac{3}{5}}$ ③

(6) 7. Find the differential of the function $y = \left(\frac{x+1}{x-1}\right)^6$.

$$dy = 6\left(\frac{x+1}{x-1}\right)^5 \frac{(x-1)1-(x+1)\cdot 1}{(x-1)^2}\,dx$$

④ for derivative with rule *

-2pts for missing dx

$$= -12\left(\frac{x+1}{x-1}\right)^5 \frac{1}{(x-1)^2}\,dx$$

$dy = \boxed{-12\left(\frac{x+1}{x-1}\right)^5 \frac{1}{(x-1)^2}\,dx}$ $\boxed{6}$

(10) 8. (a) Find the linearization $L(x)$ of the function $f(x) = \sqrt{x}$ at $a = 1$.

$L(x) = f(1) + f'(1)(x-1)$ ③

$f(x) = \sqrt{x}$ $f(1) = 1$

$f'(x) = \frac{1}{2\sqrt{x}}$ $f'(1) = \frac{1}{2}$

③

$L(x) = \boxed{1 + \frac{1}{2}(x-1)}$ $\boxed{6}$

(b) Use a linear approximation to estimate the number $\sqrt{1.1}$.

$f(x) \approx L(x)$, for x near 1

$\sqrt{x} \approx 1 + \frac{1}{2}(x-1)$, for x near 1

$\sqrt{1.1} \approx 1 + \frac{1}{2}(1.1-1) = 1 + \frac{1}{2}(0.1) = 1.05$

NPC

$\sqrt{1.1} \approx \boxed{1.05}$ $\boxed{4}$

(6) 9. Suppose that x and y are functions of t and are related by the equation $x^2 + y^2 = 1$. If $\frac{dy}{dt} = -2$, find $\frac{dx}{dt}$ when $(x, y) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$x^2 + y^2 = 1 \longrightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ ②

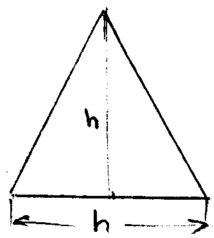
$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$

When $(x,y) = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$: $\frac{dx}{dt} = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}(-2) = 2$

④

$\boxed{2}$ $\boxed{6}$

(12) 10. Gravel is being dumped from a conveyor belt at a rate of 30 ft^3/min and its coarseness is such that it forms a pile in the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile increasing when the pile is 10 ft high?



Let V be the volume of the conical pile, r the radius of the base and h the height

Given: $\frac{dV}{dt} \overset{②}{=} 30$ ft^3/min, $h = 2r$

Find $\frac{dh}{dt}$ when $h = 10$ ft

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

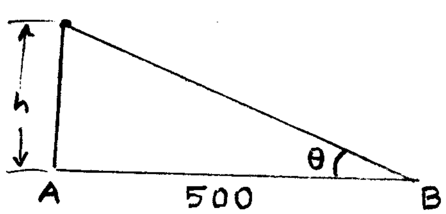
$$V = \frac{\pi}{12} h^3 \quad ④$$

$$\frac{dV}{dt} = \frac{\pi}{4} h^2 \frac{dh}{dt} \; ④ \longrightarrow \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

When $h = 10$: $\frac{dh}{dt} = \frac{4}{\pi 10^2} 30 = \frac{6}{5\pi}$ ft/min

$\boxed{② \quad \frac{6}{5\pi} \text{ ft/min}}$ $\boxed{12}$

(12) 11. A balloon was released at point A on level ground and is rising at a rate of 140 ft/min. The balloon is observed by a telescope located on the ground at point B which is 500 ft from point A. How fast is the telescope's angle of elevation changing when the balloon is 500 ft above ground?



Given $\frac{dh}{dt} \overset{②}{=} 140$ ft/min

Find $\frac{d\theta}{dt}$ when $h = 500$ ft

$$\tan\theta = \frac{h}{500} \quad ③$$

$$\sec^2\theta \cdot \frac{d\theta}{dt} = \frac{1}{500}\frac{dh}{dt} \; ④ \rightarrow \frac{d\theta}{dt} = \frac{\cos^2\theta}{500}\frac{dh}{dt}$$

When $h = 500$, $\theta = \frac{\pi}{4}$, $\cos\theta = \frac{1}{\sqrt{2}}$

$$\frac{d\theta}{dt} = \frac{\frac{1}{2}}{500} 140 = \frac{14}{100} = \frac{7}{50} \text{ rad/min}$$

$\boxed{③ \quad \frac{7}{50} \text{ rad/min}}$ $\boxed{12}$