1. (20 points) Let $X_1, X_2, X_3$ and $X_4$ be independent random variables, each uniformly distributed on the interval $(-1, 1)$. Find

(a) $P(X_1 < X_2 < X_3 < X_4)$
(b) $P(X_1^2 < (X_1 + X_2)^2)$
(c) $P(X_1^2 > X_2^2 + X_3^2)$

2. (20 points) A fair die is rolled repeatedly until it comes up ace. This procedure is repeated 100 times. Find

(a) the probability that exactly four threes are rolled in exactly 5 of the 100 repetitions;
(b) the mean and variance of the total number of threes rolled.

3. (20 points) The number of cars arriving at the McDonald’s drive-up window in a given day is a Poisson random variable, $N$, with parameter $\lambda$. The numbers of passengers in these cars are independent random variables, $X_i$, each equally likely to be one, two, three or four. Find the moment generating function of

$$S = \sum_{i=1}^{N} X_i,$$

the total number of passengers in all the cars.

4. (20 points) Let $X_1, X_2, \ldots$ be independent random variables, each uniform on the interval $(0, 1)$, and let $S_n = X_1 + X_2 + \ldots + X_n$.

(a) Find $\lim_{n \to \infty} P(S_n \leq t)$ for all $t > 0$.
(b) Find $\lim_{n \to \infty} P(S_n/n \leq t)$ for all $t > 0$.
(c) Find $\lim_{n \to \infty} P((S_n - n/2)/\sqrt{n/12} \leq t)$ for all $t > 0$.
(d) Find $\lim_{n \to \infty} P(\prod_{i=1}^{n} X_i^{1/n} \leq t)$ for all $t > 0$.
(e) Find $\lim_{n \to \infty} P(\max(X_1, \ldots, X_n) \leq t)$ for all $t > 0$. 
5. (20 points) Let \( x = (x_1, x_2, \ldots, x_n) \) be a permutation of the integers 1 to \( n \). Let \( y = (y_1, y_2, \ldots, y_n) \) be the sequence of relative ranks of the \( x_i \)'s; i.e., \( y_i = k \) if \( x_i \) is the \( k \)-th smallest of the first \( k \) \( x_i \)'s.

For example, if \( n = 5 \) and \( x = (2, 1, 4, 5, 3) \), then \( y = (1, 1, 3, 4, 3) \).

(a) If \( y = (1, 2, 1, 3, 1) \), what is \( x \)?

(b) Give a rule to compute \( x \)'s from \( y \)'s.

(c) The result of part (b) shows that, for any fixed \( n \), there is a one-to-one correspondence between the sets of all possible \( x \)'s and all possible \( y \)'s. Now suppose \( X = (X_1, X_2, \ldots, X_n) \) is a random permutation of the integers 1 to \( n \), and \( Y = (Y_1, Y_2, \ldots, Y_n) \) are the corresponding relative ranks. Show that the \( Y_i \)'s are independent, with each \( Y_i \) uniformly distributed on the integers 1 to \( i \).