25 points/problem

1. Let \( X_1, X_2, \ldots, X_{11} \) be independent identically distributed random variables each having density \( f(t) = e^{-t}, \ t > 0, \ f(t) = 0 \) elsewhere. Find the densities of the maximum of the eleven \( X_i \)'s, of the minimum of the eleven \( X_i \)'s, and of the second largest of the eleven \( X_i \)'s.

2. Let \((X, Y)\) be uniformly distributed with on the circumference of the unit square \( \{(x, y) : |x| = 1 \text{ or } |y| = 1\} \). (So, for example, \( P(a < X < b \text{ and } Y = 1) = (b - a)/8 \) if \(-1 \leq a < b \leq 1\). Find the distributions of \( X + Y \) and \( XY \). Are \( X \) and \( Y \) independent? Find the joint density of \((X, Y)\), where \( \Theta \) is a random rotation of a point about \((0, 0)\) through an angle uniformly distributed on \([0, 2\pi] \).

3. Let \( X_1, \ldots, X_{100} \) be independent normal \((0, 1)\) variables. Let \( Y_1, \ldots, Y_{100} \) be independent variables each with distribution \( P(Y_i = 1) = P(Y_i = -1) = \frac{1}{2} \). Let \( Z_1, \ldots, Z_{100} \) be independent variables each with a uniform distribution on \((-1, 1)\). Suppose all the 300 \( X \)'s, \( Y \)'s, \( Z \)'s are independent.

   i) Is \( \sum_{i=1}^{100} X_i \) exactly normal? If not, why not?

   ii) Is \( \sum_{i=1}^{100} Y_i \) exactly normal? If not, why not?

   iii) Is \( \sum_{i=1}^{100} Z_i \) exactly normal? If not, why not?

   iv) Estimate \( P(\sum_{i=1}^{100} X_i Y_i Z_i > 10) \), in terms of the standard normal distribution function.

   v) Find \( P(\prod_{i=1}^{100} X_i \prod_{i=1}^{100} Y_i \prod_{i=1}^{100} Z_i > 0) \).

4. A die is rolled until two sixes in a row come up. Find the expected number of rolls and the expected total of all the numbers rolled. Also find the distribution
of the number of sixes rolled and the distribution of the number of fives rolled.