QUALIFYING EXAMINATION
AUGUST 2000
MATH 519 - Prof. Studden

All problems have the same point value.

1. The annual number of accidents for an individual driver has a Poisson distribution with mean \( \lambda \). The Poisson means \( \lambda \), of a heterogeneous population of drivers, have a gamma distribution with mean 0.1 and variance 0.01. Calculate the probability that a driver, selected at random from the population, will have 2 or more accidents in one year. The gamma density is given by

\[
f(x) = \frac{1}{\theta^a} e^{-\frac{x}{\theta}} \frac{x^{a-1}}{\Gamma(a)}.
\]

2. Let \( X_n \) be any sequence of random variable such that \( \text{Var}(X_n) \leq c\mu_n \) for some fixed constant \( c \) and \( \mu_n = \text{E}X_n \to \infty \). Show that \( \lim_{n \to \infty} P(X_n > a) = 1 \) for all \( a \).

3. For any random variable \( X \) and \( Y \) determine whether the following are true or false;
   a) \( X \) and \( Y - \text{E}(Y|X) \) are uncorrelated,
   b) \( \text{Var}(Y - \text{E}(Y|X)) = \text{E}(\text{Var}(Y|X)) \),
   c) \( \text{Cov}(X,\text{E}(Y|X)) = \text{Cov}(X,Y) \).

4. An urn contains \( W \) white and \( B \) black balls. Balls are randomly selected without replacement from the urn until \( W \) white balls have been removed. (1 \( \leq W \)).
   a) If \( X \) is the number of black balls that are selected, what is \( P(X=k) \) (\( k = 0, 1, \ldots, B \)).
   b) What is \( \text{E}(X) \)?

5. Let \( S_n = X_1 + \cdots + X_n \) where the \( X_i \) are independent and uniformly distributed on \((0,1)\).
   a) What is the moment generating function of \( S_n \)?
   b) Show that \( f_n(x) = F_{n-1}(x) - F_{n-1}(x-1) \) where \( f_k \) and \( F_k \) are the density and distribution function of \( S_k \) respectively.
   c) Show by induction that

\[
f_n(x) = \frac{1}{(n-1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (x-k)^{n-1}.
\]
   d) Obtain the moment generating function of \( S_n \) directly from the density in part c.

6. Let \( X_1, \ldots, X_n \) be independent random variables with common distribution which is uniform on the interval \((-1/2, 1/2)\). Show that the random variables

\[
Z_n = \sqrt{n} \frac{\sum_{i=1}^{n} X_i}{\sum_{i=1}^{n} X_i^2}
\]

converge in distribution to some random variable \( Z \) and identify the distribution of \( Z \).

7. Let \( X_1, X_2, X_3 \) be independent normal random variables with mean zero and variance one. What is the distribution of

\[
\frac{X_1 + X_2X_3}{\sqrt{1 + X_3^2}}.
\]