MA 519 Introduction to Probability  
January 2000, Qualifying Examination

Instructor: Yip

• This qualifying exam contains five questions.
• By all means simplify your answers as much as possible.
• It might be useful to know that for any positive integers $m$ and $n$,
  \[ \int_0^1 x^m (1 - x)^n \, dx = \frac{m! n!}{(m + n + 1)!} \]
• A normal table is provided at the end.

1. There are $n$ people among whom are $A$ and $B$. They stand in a row randomly. What is the probability that there will be exactly $r$ people between $A$ and $B$? What is the corresponding probability if they stand in a circular ring? (In this case, consider only the arc going from $A$ to $B$ in the positive (i.e. counter-clockwise) direction.)

2. Consider a large collection of coins such that the probability $p$ of a coin giving a head is itself a random variable which is uniformly distributed in $[0, 1]$. Let $X$ be the total number of heads in $n$ tossing of the coins. Find $P(X = i)$ ($i = 0, 1, \ldots, n$) in the following two situations:
   (a) Pick a coin at random and then toss this coin $n$ times.
   (b) Pick a coin at random for each tossing.

3. Consider a sequence of Bernoulli trials of tossing a coin with $p$ as the probability of giving a head. Let $X$ be the number of trials for the $m$-th head to occur. Find the moment generating function $M_X(s)$ of $X$.
   (Note: Given any positive random variable $X$, its moment generating function $M_X(s)$ is defined as $E(e^{-sX})$.)

4. Let $X$ and $Y$ be two independent, identically and exponentially distributed random variables:
   \[ P(X \in (x, x + dx)) = \lambda e^{-\lambda x} \, dx, \quad x \geq 0 \]
   \[ P(Y \in (y, y + dy)) = \lambda e^{-\lambda y} \, dy, \quad y \geq 0 \]
   Let $T_1 = \min(X, Y)$, $T_2 = \max(X, Y)$ and $W = T_2 - T_1$. 

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(a) Find the probability density functions of $T_1, T_2$ and $W$.
(b) Find the joint probability density function of $T_1$ and $W$.
(c) Are $T_1$ and $W$ independent?

5. There are 100 light bulbs whose lifetimes $T$’s are independent exponentials with mean 5 hours (i.e. the probability density function of $T$ is $\frac{1}{5} e^{-\frac{t}{5}}$ for $t \geq 0$). If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one. In addition, it takes a random time, uniformly distributed over $(0, 0.5)$ to replace a failed bulb. Approximate the probability that all bulbs have failed by time 550?