QUALIFYING EXAMINATION
AUGUST 2002
MATH 519 - Prof. Ma

1. Tom and Bob are playing a game: each one picks a number randomly from the interval $[0,1]$. Denote Tom's number by T and Bob's number by B. Then they solve a quadratic equation $\lambda^2 - 2T\lambda + B = 0$, and Tom wins if and only if there are two distinct real solutions. Who is more likely to win? What is the absolute value of the difference of the winning probabilities of Tom and Bob?

2. Let X be a standard normal random variable ($\sim N(0,1)$), and Y is a random variable that takes two values 1 and -1 with equal probability 1/2 and is independent of X. Answer the following questions and justify your answers:

   (i) Is the random variable $Z = XY$ a normal random variable?

   (ii) Are X and Z correlated?

   (iii) Is (X, Z) a 2-dimensional Gaussian vector?

3. Suppose that U is a standard Cauchy random variable, that is, it's density function is $f_U(x) = \frac{1}{\pi}\frac{1}{1+x^2}$.

   (i) What is the law of $1/U$?

   (ii) Show that for any $\varepsilon \in [0,1]$, it holds that $E[|U|^{1-\varepsilon}] \geq 1$.

   defined F_X,

4. Let $X_1, X_2, \cdots$ be a sequence of independent Poisson random variables with parameters $\theta_1, \theta_2, \cdots$, respectively.

   (i) Suppose that $m \geq 1$, determine the distribution of $X_1 + \cdots + X_m$;

   (ii) Suppose that $\theta_i = 1$, $\forall i$, and define $\xi_n = \prod_{k=1}^n (1 + X_k)$. Show that there exists a real number ξ such that

   $$P\left\{\frac{\ln \xi_n}{n} \to \xi\right\} = 1.$$

   Determine the number ξ as well.

   (iii) Suppose again $\theta_i = 1$, $\forall i$, and define $\eta_n = \prod_{k=1}^n X_k$. Show that for any $\varepsilon > 0$, $\lim_{n\to\infty} P\{|\eta_n| > \varepsilon\} = 0$ (that is, the sequence $\{\eta_n\}$ converges to 0 *in probability*).

   (iv) Is it possible to find a random variable η such that it has finite expectation, and that $\lim_{n\to\infty} E|\eta_n - \eta| = 0$?

5. Suppose that $\{Y_n\}$ is a sequence of random variables such that for each n, Y_n is Poisson with parameter n.

   (i) Analyze the convergence of Y_n/n as $n \to \infty$ and determine the limit, if any.

   (vi) Analyze the convergence of the sequence $\frac{Y_n - n}{\sqrt{Y_n}}$, $n \geq 1$, and determine the law of the limit, if it exists.

6. Let $U_1, U_2, \cdots$ be an i.i.d. sequence of uniform random variables on $[0,1]$. Define a sequence of random variables $\{V_n\}$ recursively as follows:

$$V_1 = U_1, \qquad V_n = \begin{cases} 2V_{n-1}U_n, & \text{if } V_{n-1} \in [0, 1/2] \\ \\ (2V_{n-1} - 1)U_n, & \text{if } V_{n-1} \in [1/2, 1]. \end{cases}$$

   (i) Calculate $E\{V_n | V_{n-1}\}$ for $n \geq 2$.

   (ii) Show that the probabilities $P\{V_n < 1/2\}$ converges to a number $0 \leq a \leq 1$ as $n \to \infty$; and determine the number a.