MATHEMATICS QUALIFYING EXAMINATION
JANUARY 2007
MATH 519 - Prof. Sellke

Each problem is worth 20 points.

1. Twelve dots are arranged in four rows, with three dots in each row. Randomly choose four of the twelve dots. Let $N$ be the number of rows with no chosen dot. Find the mean and variance of $N$.

2. Let $X_1$ and $X_2$ be random variables with joint density
$$f_{X}(x_1, x_2) = \begin{cases} 
3x_1 & \text{if } 0 < x_2 < x_1 < 1 \\
0 & \text{else}
\end{cases}$$

Let $Y_1 = \frac{1}{X_1}$ and $Y_2 = \frac{1}{X_2}$. Find the joint density $f_{Y}(y_1, y_2)$ of $Y_1$ and $Y_2$.

3. Suppose that a solution now contains a single living bacterium. This organism has the property that, after 24 hours, it will give rise to a random number $N_1$ of descendants with a Geometric($p$) “number of failures” distribution:
$$P\{N_1 = k\} = q^k p, \quad k = 0, 1, 2, \ldots$$

with $p \in (0, 1)$ and $q = 1 - p$ and $EN_1 = \frac{q}{p}$. (So, $N_1$ is the population size after 24 hours.) Furthermore, each bacterium present in 24 hours will give rise after another 24 hours to a random number of descendants with the same Geometric($p$) “number of failures” distribution, with different bacteria having independent numbers of descendants.

Let $N_2$ be the population size 48 hours from now. Find $E(N_1 | N_2 = 0)$.

(Hint: The maximum value of this quantity as $p$ varies between 0 and 1 is $\frac{1}{3}$.)

4. Let $X$ and $Y$ be independent standard normal (i.e., $N(0, 1)$) random variables. Find $P\{3X^2 < Y^2\}$.

5. Let $U_1, U_2, \ldots, U_n$ be iid $U[0, 1]$ random variables, with order statistics
$$0 \leq U_{(1)} \leq U_{(2)} \leq \cdots U_{(n)} \leq 1.$$ For $k = 1, 2, \ldots, n + 1$, let $G_k = U_{(k)} - U_{(k-1)}$ be the length of the $k^{th}$ “gap” (where we set $U_{(0)} = 0$ and $U_{(n+1)} = 1$). Let
$$L_n = \max\{G_k, \quad 1 \leq k \leq n + 1\}$$
be the length of the largest gap.

When \( n = 10^{43} \), the median of the random variable \( L_n \) is approximately an integer power of \( \frac{1}{10} \), so that

\[
\text{median}(L_{10^{43}}) \approx 10^{-j}
\]

for some integer \( j \).

Find \( j \), and justify your answer. (Heuristic reasoning is fine.)