MATHEMATICS QUALIFYING EXAMINATION

JANUARY 2007

MATH 519 - Prof. Sellke

Each problem is worth 20 points.

1. Twelve dots are arranged in four rows, with three dots in each row. Randomly choose four of the twelve dots. Let N be the number of rows with no chosen dot. Find the mean and variance of N.
2. Let X_1 and X_2 be random variables with joint density
$$f_X(x_1, x_2) = \begin{cases} 3x_1 & \text{if} \quad 0 < x_2 < x_1 < 1 \\ 0 & \text{else} \end{cases}$$
Let $Y_1 = \frac{1}{X_1}$ and $Y_2 = \frac{1}{X_2}$. Find the joint density $f_Y(y_1, y_2)$ of Y_1 and Y_2.
3. Suppose that a solution now contains a single living bacterium. This organism has the property that, after 24 hours, it will give rise to a random number N_1 of descendants with a Geometric(p) "number of failures" distribution:
$$P\{N_1 = k\} = q^k p, \quad k = 0, 1, 2, \ldots$$
with $p \in (0, 1)$ and $q = 1 - p$ and $EN_1 = \frac{q}{p}$. (So, N_1 is the population size after 24 hours.) Furthermore, each bacterium present in 24 hours will give rise *after another* 24 *hours* to a random number of descendants with the same Geometric(p) "number of failures" distribution, with different bacteria having independent numbers of descendants.

   Let N_2 be the population size 48 hours from now. Find $E(N_1 | N_2 = 0)$. (Hint: The maximum value of this quantity as p varies between 0 and 1 is $\frac{1}{3}$.)
4. Let X and Y be independent standard normal (i.e., $N(0, 1)$) random variables. Find $P\{3X^2 < Y^2\}$.
5. Let $U_1, U_2, \ldots, U_n$ be iid $U[0, 1]$ random variables, with order statistics
$$0 \le U_{(1)} \le U_{(2)} \le \cdots U_{(n)} \le 1.$$
For $k = 1, 2, \ldots, n+1$, let $G_k = U_{(k)} - U_{(k-1)}$ be the length of the k^{th} "gap" (where we set $U_{(0)} = 0$ and $U_{(n+1)} = 1$). Let
$$L_n = \max\{G_k, \quad 1 \le k \le n+1\}$$

be the length of the *largest* gap.

When $n = 10^{43}$, the median of the random variable L_n is approximately an integer power of $\frac{1}{10}$, so that

$$\text{median}(L_{10^{43}}) \approx 10^{-j}$$

for some integer j.

Find j, and justify your answer. (Heuristic reasoning is fine.)