QUALIFYING EXAMINATION

August 2009

MA 519 (M. D. Ward)

1. Consider $n \geq 2$ random points, placed uniformly and independently, onto the edge of a circle with circumference 1. [An "arc" denotes a path on the circle.]

1a. (5 pts) Find the density of the length of the arc that connects the first point to the closest of the other points.

1b. (3 pts) Find the density of the straight line distance (i.e., *not* on the circle) between the two points described in part **1a**.

1c. (5 pts) Consider the length of the largest arc that contains none of the n points in its interior. Prove that the length of this largest arc goes to 0 in probability as $n \to \infty$.

2. (5 pts) Consider independent random variables X and Y, with X uniform on $(0, 2)$ and with Y uniform on $(0, 3)$. Let $M = \max(X, Y)$ and $m = \min(X, Y)$. Find $P(m^2 > M)$.

3. (5 pts) Ten students organize a tournament. Each student competes against each other student exactly once; all competitions are independent. In each competition, both students are equally-likely to win; no ties are allowed. Each student begins the tournament with 9 pennies. Each student loses a penny for each of her losses and wins a penny for each of her wins.

Find the probability that the ten students have ten distinct numbers of pennies at the end of the tournament.

4. Let Y and $X_1, X_2, X_3, \ldots$ be independent exponential($\lambda = 1$) random variables.

4a. (5 pts) Find $P(nY > X_1 + X_2 + \cdots + X_n)$, where n is a positive integer.

4b. (2 pts) Find the limit of the probability in **4a** as $n \to \infty$.

4c. (5 pts) Find $\lim_{n\to\infty} P(n + \sqrt{n}Y > X_1 + X_2 + \cdots + X_n)$.

5. (5 pts) Let Y and Z be two independent standard normal random variables. Calculate the expected value of the random variable $\max(Y, Z)$.

6. (5 pts) Consider a die with six sides: 1 side is red, 2 sides are white, and 3 sides are blue. Let X_j denote the result of the jth roll of the die; use R, W, B to denote red, white, and blue. A "run" is a sequence of consecutive rolls with the same color. E.g., in 12 die rolls,

$$X_1, X_2, \ldots, X_{12} = W, W, B, B, R, B, W, R, B, B, B, B,$$

the sequence has seven runs, made of colors white, blue, red, blue, white, red, and blue.

In a sequence of n rolls of a die, find the average number of runs.

TABLE 5.1: AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998