Math 519 – Prof. Davis
Qualifying Examination
January, 2010

(Problems 1 and 6 are twelve points each. The others are 19 points each.)

1. Toss a coin with probability p of heads repeatedly. Let X, Y, Z, and W be the tosses on which the first, second, third and fourth heads respectively appeared. For example if the tosses are $TTTHTHTTHH$ then $X = 4$, $Y = 6$, $Z = 9$, and $W = 10$. Find the probability that $Y - X > X$. Find P (either $X = 6$ or $Y = 6$ or $Z = 6|W = 15$).

2. Let X, Y, and Z be independent uniform $(0, 1)$ random variables. Find the joint density of (X, XY, XYZ).

3. Let Y_1, Y_2, and Y_3 be uniform on $(0, 1)$ and let X_1, and X_2 have density $f(t) = t/2$, $0 < t < 2$. Find the probability that the median of all the five X_i and Y_i is one of the X_i.

4. Let U_i be uniform on $(i-1, i+1)$ for $i = 1, 2, \ldots, n$ and let the U_i be independent. Let N be the number of integers i for which $U_{1-1} > U_i$. Find the mean of N. Also find the mean and variance of $\Sigma_{i=1}^{n}([U_{i-1} - U_i])^+$. ($a^+$ equals a if a is positive and equals 0 if a is not positive)

5. Two shuffled decks of one hundred cards numbered one to one hundred are dealt simultaneously one at a time. Find the probability that on none of the one hundred deals do the numbers on the two cards dealt differ by more than one. Also find the probability that it never happens that the two numbers on any of the one hundred deals differ by more than 97. Your answers to this problem do NOT have to be in the form of a single number but they should be in a form for which you could, with just a pencil, in twenty minutes or so produce a single number.

6. A point P is picked at random on the line segment $\{(x, y, z) : 0 \leq x \leq 10, y = z = 0\}$. Then this line segment is rotated, with the end at $(0, 0, 0)$ fixed, in a random three dimensional direction, so that the other end of the rotated line segment is uniformly distributed on the surface of the sphere about the origin of radius ten. Find the density $f(x, y, z)$ of the position of P after the rotation.