519 Qualifying Exam August 2013

Each problem is worth 20 points. Answers should be in a reasonably simple form.

1. An urn contains 5 balls numbered $1, 2, 3, 4, 5$. The balls are identical, aside from the numbering. Draw two balls from the urn at random, record their numbers, and return the balls to the urn. Repeat three times. Thus, you obtain four independent Simple Random Samples of size 2 from the urn. Find the numerical probability that every one of the 5 balls appears at least once in your four Simple Random Samples of size 2.

2. Let (X, Y) and (Z, W) be independent random vectors each with density uniform in the unit square $\{(s, t) : 0 < s < 1, 0 < t < 1\}$. Find the density of $(X, Y) + (Z, W)$.

3. Let X be uniform on $(2, 4)$. Let $Y = \sin(X^2)$. Let $f(y)$ be a density for Y which is as continuous as possible. Find $f(0)$.

4. Let X_i, for $1 \le i \le 5$, be independent exponential (1) random variables. Let N be the number of these random variables which are a nearest neighbor of one of the others (meaning their distance from one of the others is the least of the four distances). Find the expectation of N.

5. A standard die is rolled repeatedly until a six comes up. Each time a six does not come up a fair coin is tossed. A running tally is kept of the number of heads minus the number of tails. [For example if the rolls of the die are $3, 2, 5, 6$ and the coin tosses are H, H, T then the running tally is $1, 2, 1$] Find the probability that the absolute value of this running tally never exceeds two.