PUID Number: ____________________

MA 51900 PROBABILITY QUALIFYING EXAMINATION

January 2013

Each question is worth 10 points, so the entire exam consists of 50 points.

1. Three points are chosen independently and uniformly inside the unit square in the plane. Find the expected area of the smallest closed rectangle that has sides parallel to the coordinate axes and that contains the three points.

2. Let $X_1, X_2, X_3, \ldots, X_n$ be a sequence of independent, identically distributed Bernoulli(p) random variables. Let Y_n denote the total number of "runs" (of consecutive 0's or of consecutive 1's). Find $\mathbb{E}(Y_n)$ and $\mathrm{Var}(Y_n)$.

[E.g., if $X_1, \ldots, X_{20} = \overbrace{0}^{\text{run 1}}, \overbrace{1}^{\text{run 2}}, \overbrace{0,0,0,0,0}^{\text{run 3}}, \overbrace{1,1,1}^{\text{run 4}}, \overbrace{0,0}^{\text{run 5}}, \overbrace{1,1}^{\text{run 6}}, \overbrace{0,0,0,0}^{\text{run 7}}, \overbrace{1,1}^{\text{run 8}}$, then $Y_{20} = 8$.]

Hint: Consider using indicator random variables.

3. Consider a collection of $2n$ people: n women and their n husbands. Randomly put the people into n groups consisting of 2 people each. [A group may consist of any two people, regardless of gender.] Let $A_{j,n}$ denote the event that exactly j of the n groups consist of a married couple. Give an exact formula for $P(A_{j,n})$.

4. Assume that X_1, X_2 are IID random variables such that $\mathbf{E}\left[|X_j|^2\right] < \infty$ for $j = 1, 2$, and there exists $c > 0$ such that $(X_1 + X_2)/c$ has the same probability distribution as X_1.

a. Compute c.

b. Prove that X_1 is normally distributed.

5. Let X be uniformly distributed on the interval $(0, 1)$. Consider a sequence $\{Y_k\}_{k\geq 1}$ of independent random variables that each have density $1/(1+y)^2$ for $y > 0$. Also assume that the Y_k are independent of X. Let N be the index of the first Y_k which is strictly larger than X, and for succinctness, define $Z := Y_N$. Find the joint probability density function of X and Z.