Probability Qualifying Exam

January 2014

1. Roll a fair, six-sided die 10 times. Find the numerical probability that each side appears at least once.
2. Let X be the sample median of 1001 *iid* $\mathrm{U}[0,1]$ random variables. Find a good numerical approximation for the 75th percentile of the distribution of X.
3. A box contains 100 white balls and 3 black balls, identical except for color. Draw balls randomly from the box, one-by-one and without replacement. Let N be the number of the draw on which the first black ball is drawn. (So, for example, $N = 2$ if the first draw is white and the second draw is black.) Find the mean and variance of N.
4. Let X and Y be independent Exponential random variables with mean 1. Let

   $U = \exp(X) + 2\exp(Y)$ $\qquad\qquad$ $V = 2\exp(X^2) + \exp(Y^2)$

   and let $g(u,v)$ be a joint density for U and V which is as continuous as possible.

   Find the value of $g(3e,3e)$.
5. Customers arrive at a single-server queue according to a Poisson process with rate 1. They are served in order of arrival, but any customer who arrives when there are already two customers present (one being served and one waiting for service) departs immediately without being served. Suppose that all customers require exactly one time unit of service.
   a. For what fraction of the time is the queue empty, with no customers present?
   b. What fraction of customers leave without being served?
6. Customers enter a certain store according to a Poisson process, with rate one per minute, during the time when the store is open. Customers stay in the store for an amount of time which is uniformly distributed between 0 and 10 minutes (unless they're in the store at closing time, when all remaining customers are kicked out). The store is open from 9am until 9pm each day. What is the expected number of minutes after 9am until the first time that a customer leaves the store (or until 9pm, in the extreme unlikely event that there are no customers all day)? Give a numerical value.
7. Let $X_1, X_2, \ldots$ be *iid* $U[0,2]$ random variables, and let $\bar{X}_n = \sum_{i=1}^{n} X_i / n$. Let $Y_n = (\bar{X}_n)^3$.

   Find constants a and b so that $a\sqrt{n}\,(Y_n - b)$ converges in distribution to a limiting distribution with variance 1, and identify that distribution.