1. If ρ_0 denotes the maximum density of cars on a highway (i.e., under bumper-to-bumper conditions), then a reasonable model for traffic density ρ is given by [20pt]

$$\rho_t + (F(\rho))_x = 0, \quad F(\rho) = c\rho(1 - \rho/\rho_0),$$

where $c > 0$ is a constant (free speed of highway).

(a) (Traffic jam.) Suppose the initial density is

$$\rho(x,0) = \begin{cases} \frac{1}{2}\rho_0 & x < 0 \\ \rho_0 & x > 0. \end{cases}$$

Find the weak solution and describe the shock wave, if any.

(b) (Long red light turns green.) Find the solution for the initial data

$$\rho(x,0) = \begin{cases} \rho_0, & x < 0, \\ 0, & x > 0. \end{cases}$$

In particular, find the density $\rho(0,t)$ for $t > 0$.

[*Note:* The shock-curve solution is non-physical in this case. Find the rarefaction wave solution that is constant on the rays $x/t = v \in (-c, c)$]

2. Let u be a harmonic function in an open set $\Omega \subset \mathbb{R}^n$ and $0 \in \Omega$. Then u is real analytic in Ω and we can write it as a uniformly convergent power series near the origin: [20pt]

$$u(x) = \sum_{\alpha \in \mathbb{Z}_+^n} a_\alpha x^\alpha, \quad |x| < \delta.$$

For $k \in \mathbb{Z}_+$ let $p_k(x) = \sum_{|\alpha|=k} a_\alpha x^\alpha$.

(a) Prove that p_k is a k-homogeneous *harmonic* polynomial:

$$x\nabla p_k(x) = k\, p_k(x), \quad \Delta p_k(x) = 0.$$

(b) Prove that p_l and p_m are orthogonal on the unit sphere; i.e,

$$\int_{\partial B_1} p_l(\theta) p_m(\theta) dS_\theta = 0, \quad l \neq m.$$

[*Hint:* Use that $\partial_\nu p_k(\theta) = k\, p_k(\theta)$ for $\theta \in \partial B_1$.]

3. Let u be a bounded solution of the heat equation [20pt]

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R}^n \times (0,\infty)$$

with the initial condition $u(x,0) = g(x)$, where $g \in C_0^\infty(\mathbb{R}^n)$. Prove that in $\mathbb{R}^n \times (0,\infty)$

$$|D_x^\alpha D_t^j u(x,t)| \to 0 \quad \text{as } |x| + t \to \infty$$

for any multiindex $\alpha \in \mathbb{Z}_+^n$ and $j \in \mathbb{Z}_+$.

[*Hint:* Let Φ be the fundamental solution of the heat equation. Start by showing that $\Phi(x,t) \to 0$ as $|x| + t \to \infty$. Then consider the cases $\alpha = 0$, $j = 0$; α arbitrary, $j = 0$; α, j arbitrary.]

4. Let u be a *positive* harmonic function in the upper half space $H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$. [20pt]
Prove that if $e \in H$ and u is bounded on the ray $\{\lambda e : \lambda > 0\}$, then u is bounded in every cone $\Gamma_\alpha = \{x_n > (\cos\alpha)|x|\}$ for any $0 < \alpha < \pi/2$.

[*Hint:* Consider the family of rescalings $u_\lambda(x) = u(\lambda x)$, $\lambda > 0$.]

5. Let $U = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_i > 0, i = 1, 2, 3\}$ be the positive octant in $\mathbb{R}^3$ and consider the following initial-boundary value problem for the wave equation [20pt]

$$\begin{aligned} u_{tt} - \Delta u = 0 \quad & \text{in } U \times (0, \infty) \\ u = 0 \quad & \text{on } \partial U \times (0, \infty) \\ u = g,\ u_t = h \quad & \text{on } U \times \{0\}. \end{aligned}$$

Suppose g and h are C^∞ functions supported in the ball $B_\delta(1, 1, 1)$ for small $\delta > 0$. Find the smallest time T that will guarantee that

$$u(1, 1, 1, t) = 0 \quad \text{for } t \geq T.$$

[*Hint:* You may want to use the reflection principle.]