MA523 – Qualifying Exam, January 2016

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Instruction: Please provide complete and detailed solutions to each problem that you work. More weight will be given to a complete solution of one problem than solutions of the easy bits from two different problems.

1. (20 points) Find an explicit solution of the Cauchy problem

$$\begin{cases} u_x + 2xu_y = u^2, \\ u(1, y) = y^3, \end{cases}$$

whose domain includes a neighborhood of the line $x = 1$.

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2. (20 points) a) For any open ball $B(0,R) = \{x \in R^3 : |x| < R\}$, apply the maximum principle of Laplace equation on spherical shell domains to show that there exists at most one solution $u \in C^2(\mathbb{R}^3 \setminus B(0,R))$ to

$$\begin{cases} \Delta u = 0, & x \in \mathbb{R}^3 \setminus \overline{B(0,R)}, \\ u = 1, & x \in \partial B(0,R), \\ \lim_{|x| \to \infty} u(x) = 0. & \end{cases}$$

b) Find an explicit solution u to the above problem (*Hint*: a) implies that u must be a radially symmetric function, i.e., $u(x) = u(|x|)$).

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3. (20 points) Suppose that $u \in C^2(\mathbb{R}^n \times [0,\infty))$ solves the heat equation on $\mathbb{R}^n$:

$$\begin{cases} u_t - \Delta u = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), & x \in \mathbb{R}^n, \end{cases}$$

where $f \in C^\infty(\mathbb{R}^n)$ has compact support. Use the representation formula of u via the fundamental solution of the heat equation to show the following estimates: it holds

a) for any $k \geq 0$, $|\nabla^k u(x,t)| \leq \max_{y \in \mathbb{R}^n} |D^k f(y)|$, for any $x \in \mathbb{R}^n$ and $t > 0$.

b) $|\nabla u(x,t)| \leq C t^{-\frac{n+2}{2}} \int_{\mathbb{R}^n} |f(y)| \, dy$, for any $x \in \mathbb{R}^n$ and $t > 0$.

c) $|\nabla u(x,t)| \leq C t^{-\frac{1}{2}} \max_{y \in \mathbb{R}^n} |f(y)|$, for any $x \in \mathbb{R}^n$ and $t > 0$.

Here $C > 0$ depends only on n.

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4. (20 points) For a bounded, smooth domain $\Omega \subset \mathbb{R}^n$, a nonzero function $g \in C_0^\infty(\Omega)$, and $0 < T \leq \infty$, assume that $u \in C^2(\overline{\Omega} \times [0,T))$ solves

$$\begin{cases} u_t - \Delta u = \lambda(t) u, & \text{in } \ \Omega \times (0,T), \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \ \partial\Omega \times [0,T), \\ u = g, & \text{on } \ \Omega \times \{t = 0\}, \end{cases}$$

for some $\lambda \in C([0,T))$, where ν is the outward unit normal of $\partial\Omega$. Show that

$$\int_\Omega u^2(x,t)\,dx = \int_\Omega g^2(x)\,dx, \text{ for all } 0 \leq t < T,$$

if and only if

$$\lambda(t) = \frac{\int_\Omega |\nabla u|^2(x,t)\,dx}{\int_\Omega g^2(x)\,dx}, \text{ for all } 0 \leq t < T.$$

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5. (20 points) For any given odd function $\phi \in C^\infty(\mathbb{R})$, assume that $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ solves the Cauchy problem of the wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, \ (x,t) \in \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = \frac{\phi(|x|)}{|x|}, \ 0 \neq x \in \mathbb{R}^3; \ u(0,0) = \phi'(0), \\ u_t(x,0) = 0, \ x \in \mathbb{R}^3. \end{cases}$$

Show that

$$u(x,t) = \frac{\phi(|x|+t) + \phi(|x|-t)}{2|x|}, \ (x,t) \in \mathbb{R}^3 \times [0,\infty).$$

(*Hint*: First, by the uniqueness, $u(x,t)$ is spherically symmetric in x. Set $v(r,t) = ru$, where $r = |x|$. Then v solves the wave equation in $\mathbb{R}_+ \times (0,\infty)$.)

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