QUALIFYING EXAMINATION
AUGUST 1994
MATH 523

1. Consider the quasi-linear first order PDE in the unknown z and independent variables x and y,

$$P(x,y,z)z_x + Q(x,y,z)z_y = R(x,y,z)$$

   (a) Show that the vector field (P, Q, R) is everywhere tangent to the graph of a solution of the equation.
   (b) State carefully the general initial value problem for the equation.
   (c) State the existence and uniqueness theorem for this initial value problem.

2. Consider the first order linear PDE

$$xu_y - yu_x - u = 0$$

   in the unknown u and independent variables x and y.
   (a) Find the characteristic curves.
   (b) Express the equation in polar coordinates r and θ.
   $(x = r\cos\theta,\ \ y = r\sin\theta;\ \ u_\theta =?)$
   (c) Is there a nontrivial solution of the equation in the annular domain $1 < r < 2$? Explain.
   (d) Consider the initial value problem for the equation, with initial condition

$$u(x,0) = x^2, \quad x > 0$$

   Use a theorem to conclude that there exists a unique solution of this problem in a neighborhood of every point of the initial curve $y = 0, \quad x > 0$.
   (e) Find the solution of the initial value problem described in (d), in a neighborhood of the point $(x, y) = (1, 0)$.

3. Suppose that u and v are harmonic functions in a neighborhood of $x^0 \in \mathbb{R}^n$, and suppose that

$$D^\alpha u(x^0) = D^\alpha v(x^0),$$

   for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \geq 0$. Is it necessarily true that $u = v$ in a neighborhood of x^0? Justify your answer.

4. Suppose that $u(r,\theta)$ (polar coordinates) is harmonic in the unit ball $r < 1$, $0 \le \theta \le 2\pi$. For each assertion below, decide whether or not it is always true, and justify your answer.

(a) $\displaystyle\int_0^{2\pi} u(r_1,\theta)d\theta = \int_0^{2\pi} u(r_2,\theta)d\theta \quad \text{for all} \quad r_1, r_2;\ 0 < r_1 < 1,\ 0 < r_2 < 1.$

(b) $\displaystyle\int_0^{2\pi} \frac{\partial u}{\partial r}(r_1,\theta)d\theta = \int_0^{2\pi} \frac{\partial u}{\partial r}(r_2,\theta)d\theta \quad \text{for all} \quad r_1, r_2;\ 0 < r_1 < 1,\ 0 < r_2 < 1.$

5. Prove the following theorem on conservation of energy for the wave equation: Let $u(x,t)$ be the solution of the initial value problem

$$\begin{aligned} u_{x_1x_1} + \cdots + u_{x_nx_n} - u_{tt} = 0; &\quad x \in \mathbb{R}^n,\ 0 < t \\ u(x,0) = \varphi(x); &\quad x \in \mathbb{R}^n \\ u_t(x,0) = \psi(x); &\quad x \in \mathbb{R}^n \end{aligned}$$

and suppose that u is in C^2 for $x \in \mathbb{R}^n$ and $t \ge 0$. Suppose also that φ and ψ vanish outside some ball $B(0,R)$ in $\mathbb{R}^n$. Show that, for every $T \ge 0$,

$$\int_{\mathbb{R}^n} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2)|_{t=T}\, dx = \int_{\mathbb{R}^n} (u_{x_1}^2 + \cdots + u_{x_n}^2 + u_t^2)|_{t=0}\, dx$$

6. Consider the initial value problem

$$\begin{aligned} u_{xx} - \frac{1}{c^2}u_{tt} = 0; &\quad -\infty < x < \infty, \quad 0 \le t \\ u(x,0) = \varphi(x); &\quad -\infty < x < \infty \\ u_t(x,0) = \psi(x); &\quad -\infty < x < \infty \end{aligned}$$

where φ and ψ are given C^2 functions and c is a positive constant.

(a) Give the formula for the solution of the problem.
(b) Verify your solution by direct substitution.

7. (a) Let $u(x,t)$ be the solution of the following initial-boundary value problem for the nonhomogeneous heat equation

$$u_t - u_{xx} = f(x,t); \quad 0 < x < L,\ 0 < t \tag{1}$$
$$u(x,0) = 0; \quad 0 \le x \le L \tag{2}$$
$$u(0,t) = 0,\ u(L,t) = 0; \quad 0 \le t \tag{3}$$

where $f(x,t)$ is a given continuous function such that $f(0,t) = f(L,t) = 0$ for all $t \ge 0$. For each $\tau \ge 0$, let $v(x,t;\tau)$ be the solution of the associated pulse problem

$$v_t - v_{xx} = 0; \quad 0 < x < L,\ \tau < t \tag{4}$$

$$v(x, \tau; \tau) = f(x, \tau); \quad 0 \le x \le L \tag{5}$$

$$v(0, t; \tau) = 0, v(L, t; \tau) = 0; \quad \tau \le t \tag{6}$$

Show that (Duhamel's principle)

$$u(x,t) = \int_0^t v(x, t; \tau) d\tau$$

(b) Find the solution of problem (1), (2), (3) if

$$f(x,t) = \sin t \sin \frac{\pi x}{L}$$

(You may leave your answer in a form involving an integral in τ.)