QUALIFYING EXAMINATION

AUGUST 1995

MATH 523

1. Consider the quasilinear equation

$$zz_x + y^2 z_y = x$$

   (a) Find the general integral of the equation.

   (b) Find a solution of the equation.

2. Consider the initial value problem in two variables x, y,

$$\begin{aligned} u_x + y_y - u = 0, &\quad (x, y) \in \mathbb{R}^2 \\ u(x, -x) = x^2, &\quad x \in \mathbb{R} \end{aligned}$$

   (a) Use a theorem to show that there is a unique solution in a neighborhood of every point of the initial curve $y = -x$.

   (b) Introduce new coordinates ξ and η in terms of which the partial differential equation becomes an ordinary differential equation. Also express the initial condition in terms of the new coordinates.

   (c) Solve the transformed problem and obtain the solution of the original problem.

3. Suppose that the second order linear PDE in two independent variables,

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu + g = 0$$

   is elliptic in $\mathbb{R}^2$. Prove that the equation has no characteristic curves.

4. (a) Suppose that $u(\vec{r})$ is harmonic in $\mathbb{R}^n$ and that $u(\vec{r}) \to 0$ as $|\vec{r}| \to \infty$. Is it necessarily true that $u(\vec{r}) \equiv 0$. Justify your answer.

   (b) Let Ω be a bounded normal domain in $\mathbb{R}^n$ and suppose that $u \in C^0(\overline{\Omega})$ is a solution of the Dirichlet problem

$$\triangle u = -1 \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

   Does u attain its minimum on the boundary $\partial\Omega$? Justify your answer.

5. (a) State carefully the theorem on the domain of dependence inequality for the wave equation in two space variables

$$u_{x_1x_1} + u_{x_2x_2} - u_{tt} = 0.$$

(b) Let $u(x_1, x_2, t)$ be a C^2 solution of the initial-boundary value problem

$$\begin{aligned} u_{x_1x_1} + u_{x_2x_2} - u_{tt} &= 0; \quad -\infty < x_1 < \infty, \quad 0 < x_2 < \infty, \quad 0 \le t \\ u(x_1, 0, t) &= f(x_1, t); \quad -\infty < x_1 < \infty, \quad 0 \le t \\ u(x_1, x_2, 0) &= 0; \quad -\infty < x_1 < \infty, \quad 0 < x_2 < \infty \\ u_t(x_1, x_2, 0) &= 0; \quad -\infty < x_1 < \infty, \quad 0 < x_2 < \infty. \end{aligned}$$

Use the theorem in part (a) to prove that $u(x_1, x_2, t) = 0$ for all (x_1, x_2, t) such that $-\infty < x_1 < \infty, \quad 0 < x_2 < \infty, \quad 0 \le t < x_2$.

6. Consider the initial-boundary value problem for the heat equation

$$\begin{aligned} u_t - u_{xx} &= 0; \quad 0 < x < L, \quad 0 < t \\ u_x(0, t) = 0, \quad u(L, t) &= 0; \quad 0 \le t \\ u(x, 0) &= x^2 - 1; \quad 0 \le x \le L. \end{aligned}$$

(a) Assuming that u is of class C^2 in the closed strip $0 \le x \le L$, $0 \le t$, prove uniqueness of solution of the problem.

(b) Use the method of separation of variables to obtain the series solution of the problem. You do not have to evaluate the coefficients of the series.