1. Consider the initial value problem

\[(1 - z^3)z_x + z_y = 0\]
\[z(x,0) = f(x)\]

where \(f \in C^1(\mathbb{R})\).

(a) Write down the equation that implicitly defines the solution \(z\) near the \(x\)-axis.

(b) From your equation in (a), find formulas for \(z_x\) and \(z_y\) and verify that the PDE is satisfied.

(c) If \(f(x) = -x^3\), decide whether or not shocks ever develop for \(y \geq 0\), and justify your answer.

2. Let \(C\) be the parabola \(y = x^2\) and consider the initial value problem

\[x^2 u_x - y^2 u_y + \cos(x - y)u = e^{xy}\]
\[u|_C = \varphi\]

where \(\varphi\) is a given continuous function defined on \(C\). Prove that this problem has a unique solution in a neighborhood of the point \((1,1)\).

3. (a) State carefully the maximum principle for harmonic functions.

(b) Let \(\Omega\) be the upper-half of the unit ball in \(\mathbb{R}^3\)

\[\Omega = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 < 1, \quad z > 0\}\]

and let \(u = 1/r\) where \(r = (x^2 + y^2 + z^2)^{1/2}\). Show by direct computation that \(u\) is harmonic in \(\Omega\).

(c) Does \(u\) attain its maximum value in \(\Omega\)? Does your answer contradict the maximum principle? Explain.
4. Let $\Omega$ be a domain in $\mathbb{R}^3$

(a) Define carefully the Green’s function $G(\vec{r}', \vec{r})$ for the Dirichlet problem for $\Omega$.

(b) Write down the formula for the solution of the Dirichlet problem

$$\Delta u = 0 \quad \text{in} \quad \Omega$$
$$u = f \quad \text{on} \quad \partial \Omega$$

in terms of the Green’s function.

(c) Prove that $G(\vec{r}', \vec{r}) \geq 0$ for all $\vec{r}', \vec{r} \in \Omega$, $\vec{r}' \neq \vec{r}$.

5. Consider the initial value problem for the wave equation in three space variables:

$$u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} - u_{tt} = 0; \quad x \in \mathbb{R}^3, \quad t > 0$$
$$u(x, 0) = \varphi(x); \quad x \in \mathbb{R}^3$$
$$u_t(x, 0) = \psi(x); \quad x \in \mathbb{R}^3$$

(a) Write down the formula for the solution $u(x, t)$ of the problem.

(b) If $\varphi$ and $\psi$ vanish outside a ball of radius 3 centered at the origin, find the set of points in $\mathbb{R}^3$ where you are sure that $u$ vanishes when $t = 10$.

(c) If $\varphi$ vanishes everywhere in $\mathbb{R}^3$, and

$$\psi(x) = \begin{cases} 0 & \text{for } |x| < 1 \\ k & \text{for } 1 \leq |x| \leq 2 \\ 0 & \text{for } 2 < |x| \end{cases}$$

where $k$ is a constant, find $u(0, t)$ for all $t \geq 0$. (Your answer should be explicit, no integrals).
6. Let Ω be a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial \Omega$, and let $\vec{n}$ be the exterior unit normal on $\partial \Omega$.

(a) Suppose that $u(x_1, x_2, t)$ is of class $C^2$ in the closed half-cylinder

$$(x_1, x_2) \in \overline{\Omega}, \quad t \geq 0,$$

and that $u$ satisfies the heat equation

$$u_t - u_{x_1x_1} - u_{x_2x_2} = 0; \quad (x_1, x_2) \in \Omega, \quad t > 0$$

and the boundary condition

$$\frac{\partial u}{\partial n}(x_1, x_2, t) = -u(x_1, x_2, t); \quad (x_1, x_2) \in \partial \Omega, \quad t \geq 0.$$

Show that for any $T \geq 0$

$$\int_{\Omega} u^2(x_1, x_2, T)dx_1dx_2 \leq \int_{\Omega} u^2(x_1, x_2, 0)dx_1dx_2$$

(b) Consider the initial-boundary value problem

$$u_t - u_{x_1x_1} - u_{x_2x_2} = 0; \quad (x_1, x_2) \in \Omega, \quad t > 0,$$

$$\frac{\partial u}{\partial n}(x_1, x_2, t) = -u(x_1, x_2, t); \quad (x_1, x_2) \in \partial \Omega, \quad t \geq 0$$

$$u(x_1, x_2, 0) = \varphi(x_1, x_2); \quad (x_1, x_2) \in \Omega.$$

Prove uniqueness of solution of this problem in the class of functions which are $C^2$ in the closed half-cylinder $(x_1, x_2) \in \overline{\Omega}, \quad t \geq 0$.

7. For each of the PDEs below give an explicit example of a solution which is in $C^2(\mathbb{R}^2)$ but not in $C^3(\mathbb{R}^2)$, if such a solution exists.

(a) $u_{xx} + u_{yy} = 0$

(b) $u_{xx} - u_{yy} = 0$

(c) $u_{xx} - u_y = 0$. 