QUALIFYING EXAMINATION

JANUARY 1996

MATH 523

1. (a) Find the integral curve of the vector field $\vec{V}(x,y,z) = (z^2, x^2y, x^2z)$ that passes through the point $(1, 2, -1)$.

   (b) Find the general solution of the PDE in the unknown u, and independent variables x, y, z,

$$z^2u_x + x^2yu_y + x^2zu_z = 0.$$

2. Consider the initial value problem in two independent variables x, y,

$$u_x = 0, \quad (x,y) \in \mathbb{R}^2$$
$$u(x, x^2) = \sin x, \quad x \in \mathbb{R}$$

   Answer the following and justify your answer in each case.

   (a) Is there a global solution to the problem?

   (b) Is there a local solution of the problem in a (sufficiently small) neighborhood of the point $(-1, 1)$?

   (c) Is there a local solution of the problem in a (sufficiently small) neighborhood of the origin?

3. (a) If the plane $ax+by+cz = 0$ is characteristic for the equation $u_{xx}+u_{yy}-u_{zz} = 0$, prove that $u(x,y,z) = F(ax + by + cz)$ is a solution of the equation, for any function $F \in C^2(\mathbb{R}^1)$.

   (b) Find all the characteristic surfaces of $u_{xx} + 2u_{yy} + 3u_{zz} = 0$.

   (c) Find all the characteristic curves of the equation $yu_x + u_y - u = e^{xy}$.

4. Let Ω be a bounded normal domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$, and let $\vec{n}$ be the exterior unit normal vector on $\partial\Omega$. Prove uniqueness of solution of the boundary value problem for the biharmonic equation,

$$\triangle(\triangle u) = f \quad \text{in} \quad \Omega$$
$$u = g \quad \text{on} \quad \partial\Omega$$
$$\frac{\partial u}{\partial n} = h \quad \text{on} \quad \partial\Omega$$

where f, g and h are sufficiently smooth in their domains. Assume $u \in C^4(\overline{\Omega})$.

5. Suppose that u is harmonic in an open set $\Omega \subset \mathbb{R}^n$ and that, for some $x^o \in \Omega$,

$$u(x^o) = 1$$

and

$$D^\alpha u(x^o) = 0$$

for all multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| \geq 1$. What can you conclude about u in some neighborhood of x^o? Justify your conclusion.

6. (a) State the initial value problem for the wave equation $u_{xx} - u_{tt} = 0$ in one space variable.

   (b) Write down the formula for the solution $u(x, t)$.

   (c) If the initial data vanish outside a finite interval, find $\lim_{t \to \infty} u(x, t)$ for any $x \in \mathbb{R}$. Is this limit necessarily zero? Explain.

7. Find a formula for the solution of the initial-boundary value problem for the heat equation in one space variable,

$$\begin{gathered} u_t - u_{xx} = 0; \quad 0 < x < L, \quad 0 < t \\ u(0, t) = 0, \quad u(L, t) = 1; \quad 0 \leq t \\ u(x, 0) = \varphi(x); \quad 0 \leq x \leq L \end{gathered}$$

where $\varphi \in C^1$ and $\varphi(0) = 0$, $\varphi(L) = 1$.