1. Consider the initial value problem
\[ \begin{align*}
zz_x + z_y &= z \\
z(x, 0) &= 3x
\end{align*} \]
(a) Use an existence and uniqueness theorem to show that the problem has a unique solution in a neighborhood of every point of the initial curve \( y = 0 \).
(b) Solve the problem.

2. Let \( \Omega \) be a domain in \( \mathbb{R}^3 \) and \( \vec{V} = (P, Q, R) \) be a nonvanishing \( C^1 \) vector field in \( \Omega \). Suppose that \( u \in C^1(\Omega) \), \( \text{grad } u \neq \vec{0} \) in \( \Omega \), and that the level surfaces of \( u \),
\[ u(x, y, z) = c, \]
are the integral surfaces of \( \vec{V} \) in \( \Omega \). Prove that if \( C \) is the integral curve of \( \vec{V} \) passing through \( (x_0, y_0, z_0) \in \Omega \), then \( C \) must lie on the integral surface of \( \vec{V} \) passing through \( (x_0, y_0, z_0) \).

3. Prove uniqueness of solution of the initial-boundary value problem
\[ \begin{align*}
u_{xx} - u_{tt} - au_t - bu &= F(x, t); \quad 0 < x < L, \quad 0 \leq t \\
u(x, 0) &= \varphi(x), \quad u_t(x, 0) = \psi(x); \quad 0 \leq x \leq L \\
u(0, t) &= f(t), \quad u_x(L, t) = g(t); \quad 0 \leq t
\end{align*} \]
where \( a \) and \( b \) are nonnegative constants, and \( F, \varphi, \psi, f, \) and \( g \) are sufficiently smooth functions. Assume that \( u(x, t) \) is \( C^2 \) for \( 0 \leq x \leq L \) and \( 0 \leq t \).

4. Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \) and let \( \vec{n} \) be the exterior unit normal vector on \( \partial \Omega \).
(a) Define carefully the Green’s function \( G(\vec{r}', \vec{r}) \) for the Dirichlet problem for \( \Omega \).
(b) Write down the formula for the solution of the Dirichlet problem
\[ \begin{align*}
\nabla^2 u &= 0 \quad \text{in } \Omega \\
u &= f \quad \text{on } \partial \Omega
\end{align*} \]
in terms of the Green’s function.
(c) Show that for each fixed \( \vec{r} \) in \( \Omega \), \( \frac{\partial}{\partial n} G(\vec{r}', \vec{r}) \leq 0 \), for \( \vec{r}' \in \partial \Omega \).
(d) Show that for each \( \vec{r} \in \Omega \),
\[- \int_{\partial \Omega} \frac{\partial}{\partial n} G(\vec{r}', \vec{r})d\sigma = 1.\]
5. Consider the initial-boundary value problem for the heat equation,

\[ u_t - u_{xx} = 0; \quad 0 < x < L, \quad 0 < t \]
\[ u_x(0, t) = u_x(L, t) = 0; \quad 0 \leq t \]
\[ u(x, 0) = \begin{cases} 0 & \text{for } 0 \leq x < \frac{L}{2} \\ 100 & \text{for } \frac{L}{2} \leq x \leq L \end{cases} \]

(a) Find the series solution of the problem.
(b) Does the series solution converge uniformly when \( t = 0 \)? Explain.
(c) Prove that the solution is \( C^\infty \) when \( t > 0 \).

6. For each of the PDEs below, construct a solution which is in \( C^2(\mathbb{R}^3) \) but not in \( C^3(\mathbb{R}^3) \). If this is not possible, explain why.

(a) \( u_{xx} + u_{yy} - u_{zz} = 0, \quad (x, y, z) \in \mathbb{R}^3 \)
(b) \( u_{xx} + u_{yy} + u_{zz} = 0, \quad (x, y, z) \in \mathbb{R}^3 \)
(c) \( u_{xx} + u_{yy} - u_z = 0, \quad (x, y, z) \in \mathbb{R}^3 \)

7. Consider the linear first order PDE in two variables,

\[ a(x, y)u_x + b(x, y)u_y = 0 \]

where \( a \) and \( b \) are \( C^1 \) and do not vanish simultaneously. Prove that if \( C \) is a characteristic curve of the PDE, then a solution \( u(x, y) \) of the PDE must be constant on \( C \).

8. State carefully the theorem on the domain of dependence inequality for the wave equation in two space variables,

\[ u_{xx} + u_{yy} - u_{tt} = 0. \]