QUALIFYING EXAMINATION

AUGUST 1998

MATH 523 - Prof. Zachmanoglou

(20) 1. Consider the first order linear PDE

$$(*) \qquad u_x - yu_y + u = 0, \quad (x,y) \in \mathbb{R}^2.$$

(a) Find the characteristic curves of $(*)$.

(b) Make a transformation of coordinates,

$$\xi = \xi(x,y), \quad \eta = \eta(x,y)$$

to obtain the canonical form of $(*)$. Hint: Choose $\eta(x,y)$ so that its level curves are the characteristic curves of $(*)$.

(c) Solve the canonical form of $(*)$ and obtain the general solution of $(*)$.

(d) Find the solution of $(*)$ satisfying the initial condition

$$u(0,y) = y^2, \quad y \in \mathbb{R}.$$

(15) 2. Find the solution of the following Dirichlet problem for an annulus, in polar coordinates,

$$\nabla^2 u(r,\theta) = 0;\ 1 < r < 2,\ 0 \le \theta \le 2\pi$$
$$u(1,\theta) = 3 + 4\sin\theta; \quad 0 \le \theta \le 2\pi$$
$$u(2,\theta) = 5 + 6\sin\theta; \quad 0 \le \theta \le 2\pi$$

(15) 3. Consider the Neumann problem for the unit ball in $\mathbb{R}^3$, in spherical coordinates,

$$\nabla^2 u(r,\theta,\varphi) = 0,\ 0 \le r < 1,\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi,$$
$$\frac{\partial u}{\partial n}(1,\theta,\varphi) = \sin\varphi\cos\theta;\ 0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$$

where $\frac{\partial u}{\partial n}$ denotes the directional derivative of u in the exterior normal direction.

(a) Verify that the necessary condition for the existence of a solution is satisfied.

(b) Find all solutions of the problem.

[$x = r\sin\varphi\cos\theta,\ y = r\sin\varphi\sin\theta,\ z = r\cos\varphi$. The element of surface on the unit sphere is $d\sigma = \sin\varphi\, d\theta\, d\varphi$].

(15) 4. Consider the initial-boundary value problem for the wave equation in one space variable,

$$
\begin{aligned}
&u_{xx} - u_{tt} = 0; \quad 0 < x < L,\ 0 < t \\
&u(0,t) = u(L,t) = 0; \quad 0 \le t \\
&u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x); \quad 0 \le x \le L
\end{aligned}
$$

where φ and ψ are given functions. Use separation of variables and Fourier series to obtain the "formal" series solution of the problem ("formal": do not be concerned with convergence, continuity or differentiability). Make sure to give the formulas for the computation of the coefficients of the series.

(15) 5. Suppose that $f \in C^2(\mathbb{R})$ and f is periodic of period 2π. Prove that there is a constant $M > 0$, such that the Fourier coefficients of f satisfy the inequalities

$$
|a_n| \le \frac{M}{n^2},\ |b_n| \le \frac{M}{n^2};\ n = 1, 2, \ldots
$$

(20) 6. (a) Let Ω be a bounded normal domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$ and let $\vec{n}$ be the exterior unit normal on $\partial\Omega$. Suppose that $u(x,y,t)$ is of class C^2 for $(x,y) \in \overline{\Omega}$ and $0 \le t$, and satisfies the PDE

$$
u_{xx} + u_{yy} - u_t - qu = 0; \quad (x,y) \in \Omega,\ 0 < t,
$$

where q is a nonnegative continuous function for $(x,y) \in \overline{\Omega}$ and $0 \le t$, and the boundary condition

$$
\frac{\partial u}{\partial n}(x,y,t) = 0; \quad (x,y) \in \partial\Omega,\ 0 \le t.
$$

Show that for any $T \ge 0$,

$$
\iint_\Omega u^2|_{t=T}\, dx\, dy \le \iint_\Omega u^2|_{t=0}\, dx\, dy
$$

(b) Write down the most general initial-boundary value problem for which you can prove uniqueness of solution using the result in (a), and assuming sufficient differentiability.