Qualifying Examination

August 2000

MATH 523 - Prof. Sa Barreto

1) Let $\Omega \subset \mathbb{R}^n$ be an open subset. Given $x \in \Omega$ and $r > 0$, such that the ball $B(x,r)$ of center x and radius r is contained in Ω, the spherical mean $M(u,x,r)$ of a function $u \in C^0(\Omega)$ is defined by

$$M(u,x,r) = \frac{1}{\omega_n} \int_{\mathbb{S}^n} u(x + r\omega)\, d\omega.$$

a)(10) Let $u \in C^2(\Omega)$, show that if $\Delta u = 0$ in Ω then $u(x) = M(u,x,r)$ for every $x \in \Omega$ and $r > 0$, such that $B(x,r) \subset \Omega$.

b)(10) Suppose that Ω is a bounded set. Let $u, v \in C^2(\Omega)$ satisfy $\Delta u = \Delta v = 0$. Suppose that $u(x) \geq v(x)$ for every $x \in \Omega$. Show that if $u(x_0) = v(x_0)$ for a point x_0 in Ω, then $u(x) = v(x)$ for every $x \in \Omega$.

c)(5) Let $\Omega = \{x \in \mathbb{R}^3 : 1 < |x| < 3\}$. Find a function $u \in C^2(\Omega)$ such that $\Delta u = 0$ in Ω, $u(x) = 0$ if $|x| = 1$ and $u(x) = 5$ if $|x| = 3$. Justify all the steps in your solution.

2) Let $u(x,t) \in C^2(\mathbb{R}^n \times \mathbb{R}^+)$ satisfy

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+$$
$$u(x,0) = g(x), \quad D_t u(x,0) = f(x).$$

a)(10) Let $E(t) = \frac{1}{2} \int_{\mathbb{R}^n} \left(|\nabla u|^2 + |u_t|^2\right)\, dx$. Show that $E(t) = E(0)$ for every t.

a)(10) Show that if $f(x)$ is a radial function, i.e $f(x) = f(|x|)$, then the solution is also radial, i.e $u(x,t) = u(|x|,t)$.

b)(5) Suppose that n is odd and $f(x) = 0$ if $|x| > 4$. Let $K \subset \mathbb{R}^n$ be bounded set. Show that $\int_K |u(x,t)|^2\, dx = E(t)$ goes to zero as $t \to \infty$.

c)(10) Consider the question in item b, but now assume that n is even instead. Does $E(t) \to 0$ as $t \to \infty$?

3)Let $u(x,t) \in C^0(\mathbb{R}^n \times \mathbb{R}^+)$ satisfy

$$\left(\frac{\partial}{\partial t} - \Delta\right) u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+ \qquad (0.1)$$
$$u(x,0) = f(x),$$

where $f(x)$ is bounded.

a)(10) Is this a non-characteristic Cauchy problem? This is an equation of second order, so why do we only set the value of the function at $t = 0$, and not $D_t u(x,0)$ as well?

a)(5) Is there a relation between the support of $u(x,t)$, for fixed t, and the support of $f(x)$? Does it make a difference if n is even or odd?

c)(5) What happens to $u(t,x)$ if we switch t to $-t$ in (0.1)?

d)(5) Is $u(x,t)$ uniquely defined by (0.1)?

4) Let $u(x,y)$ satisfy the equation

$$x\frac{\partial u}{\partial y} - y\frac{\partial u}{\partial x} = -\frac{1}{3}u^4$$
$$u(x,0) = h(x).$$

a) (5) Write the equation in polar coordinates.

b) (10) Use a to show that

$$u(x,y) = \left[\tan^{-1}\left(\frac{y}{x}\right) + \left(h(\sqrt{x^2+y^2})\right)^{-3} - \frac{\pi}{2}\right)^{-\frac{1}{3}}.$$