1) Let $\Omega \subset \mathbb{R}^n$ be an open subset. Given $x \in \Omega$ and $r > 0$, such that the ball $B(x, r)$ of center $x$ and radius $r$ is contained in $\Omega$, the spherical mean $M(u, x, r)$ of a function $u \in C^0(\Omega)$ is defined by

$$M(u, x, r) = \frac{1}{\omega_n} \int_{S^n} u(x + r\omega) \, d\omega.$$ 

a)(10) Let $u \in C^2(\Omega)$, show that if $\Delta u = 0$ in $\Omega$ then $u(x) = M(u, x, r)$ for every $x \in \Omega$ and $r > 0$, such that $B(x, r) \subset \Omega$.

b)(10) Suppose that $\Omega$ is a bounded set. Let $u, v \in C^2(\Omega)$ satisfy $\Delta u = \Delta v = 0$. Suppose that $u(x) \geq v(x)$ for every $x \in \Omega$. Show that if $u(x_0) = v(x_0)$ for a point $x_0 \in \Omega$, then $u(x) = v(x)$ for every $x \in \Omega$.

c)(5) Let $\Omega = \{ x \in \mathbb{R}^3 : 1 < |x| < 3 \}$. Find a function $u \in C^2(\Omega)$ such that $\Delta u = 0$ in $\Omega$, $u(x) = 0$ if $|x| = 1$ and $u(x) = 5$ if $|x| = 3$. Justify all the steps in your solution.

2) Let $u(x, t) \in C^2(\mathbb{R}^n \times \mathbb{R}^+) \text{ satisfy}$

$$\left( \frac{\partial^2}{\partial t^2} - \Delta \right) u(x, t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+$$

$$u(x, 0) = g(x), \quad Du(x, 0) = f(x).$$

a)(10) Let $E(t) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla u|^2 + |u_t|^2) \, dx$. Show that $E(t) = E(0)$ for every $t$.

a)(10) Show that if $f(x)$ is a radial function, i.e $f(x) = f(|x|)$, then the solution is also radial, i.e $u(x, t) = u(|x|, t)$.

b)(5) Suppose that $n$ is odd and $f(x) = 0$ if $|x| > 4$. Let $K \subset \mathbb{R}^n$ be bounded set. Show that $\int_K |u(x, t)|^2 \, dx = E(t)$ goes to zero as $t \to \infty$.

c)(10) Consider the question in item b, but now assume that $n$ is even instead. Does $E(t) \to 0$ as $t \to \infty$?
3) Let \( u(x,t) \in C^0(\mathbb{R}^n \times \mathbb{R}^+) \) satisfy
\[
\left( \frac{\partial}{\partial t} - \Delta \right) u(x,t) = 0 \text{ in } \mathbb{R}^n \times \mathbb{R}^+
\]
\[u(x,0) = f(x),\]
where \( f(x) \) is bounded.

a) (10) Is this a non-characteristic Cauchy problem? This is an equation of second order, so why do we only set the value of the function at \( t = 0 \), and not \( D_t u(x,0) \) as well?

a) (5) Is there a relation between the support of \( u(x,t) \), for fixed \( t \), and the support of \( f(x) \)? Does it make a difference if \( n \) is even or odd?

c) (5) What happens to \( u(t,x) \) if we switch \( t \) to \( -t \) in (0.1)?

d) (5) Is \( u(x,t) \) uniquely defined by (0.1)?

4) Let \( u(x,y) \) satisfy the equation
\[
x \frac{\partial u}{\partial y} - y \frac{\partial u}{\partial x} = -\frac{1}{3} u^3
\]
\[u(x,0) = h(x),\]
a) (5) Write the equation in polar coordinates.

b) (10) Use a to show that
\[
u(x,y) = \left[ \tan^{-1} \left( \frac{y}{x} \right) + \left( h(\sqrt{x^2 + y^2})^{-3} - \frac{\pi}{2} \right)^{-\frac{1}{3}} \right].
\]