1) a) Solve
\[ \frac{\partial u}{\partial y} + u^5 \frac{\partial u}{\partial x} = 0, \]
\[ u(x,0) = f(x). \]

b) Find the lines on the plane \((x,y)\) where \(u\) is constant. Use this to examine whether \(u\) will develop shocks in the region \(y > 0\).

2) Let \(u(x,t) \in C^2(\mathbb{R}^n \times (0,\infty))\) satisfy
\[ \left( \frac{\partial^2}{\partial t^2} - \Delta \right) u(x,t) = f(x,t), \quad x \in \mathbb{R}^n, \; t > 0 \]
\[ u(x,0) = 0, \; \frac{\partial}{\partial t} u(x,0) = 0 \]
Find a formula for \(u(x,t)\) in terms of \(f\) when \(n = 2, 3\).

3) Let \(f \in C^\infty(\mathbb{R})\) and let \(u(x,t)\) be the solution to
\[ \left( \frac{\partial}{\partial t} - \Delta + f(t) \right) u(x,t) = 0, \quad x \in \mathbb{R}^n, \; t > 0, \]
\[ u(x,0) = \phi(x). \]
Find a formula for \(u\) in terms of \(f\) and \(\phi\).

4) a) Let \(\Omega \subset \mathbb{R}^n\) be an open subset and let \(S \subset \Omega\) be a real analytic hypersurface. Let \(\phi, \psi\) be real analytic functions on \(S\). Show that there exist an open set \(U \subset \Omega\), with \(S \subset U\), and a unique real analytic function \(u \in C^\omega(U)\) such that
\[ \Delta u = 0, \quad \text{in} \; U \]
(1)
\[ u|_S = \phi, \; \partial_\mu u|_S = \psi, \]
where \(\partial_\mu\) is the normal derivative to \(S\).

b) If \(\phi_n\), and \(\psi_n\) are sequences of real analytic functions converging uniformly to zero. Is it true that the solutions \(u_n\) of the problem (??) with data \(\phi_n, \psi_n\) converge uniformly to zero? Hint: Take \(S = \{y = 0\} \subset \mathbb{R}^2\), \(\phi_n(x) = 0\) and \(\psi_n(x) = \frac{1}{n} \sin(nx)\).

5) Let \(f \in C^2(\mathbb{R})\) satisfy \(f''(t) \geq 0\) for every \(t \in \mathbb{R}\) and
\[ \lim_{|t| \to \infty} \frac{f(t)}{|t|} = 0. \]
Show that \( f \) is constant. What is the geometric interpretation of this result?

Hint: Suppose that \( f'(t_0) > 0 \) for some \( t_0 \in \mathbb{R} \). Show that in this case \( f(t) - f(t_0) \geq f'(t_0)(t - t_0) \) for every \( t > t_0 \). Prove a similar estimate if \( f'(t_0) < 0 \) for some \( t_0 \).

6) The Poisson kernel of the ball \( B(x, R) = \{ y \in \mathbb{R}^n : |x - y| < R \} \) is

\[
H(x, y, \xi) = \frac{1}{R^n \omega_n} \frac{R^2 - |x - \xi|^2}{|y - \xi|^n},
\]

where \( \omega_n \) is the surface area on the sphere in \( \mathbb{R}^n \). So, if \( u \) is harmonic in \( B(x, R) \) and continuous in \( \overline{B(x, R)} \), and if \( |x - \xi| < R \), we have

\[
u(\xi) = \frac{R^2 - |x - \xi|^2}{R \omega_n} \int_{|y-x|=R} \frac{1}{|y-\xi|^n} u(y) \, d\sigma_y\]

a) Use this to show that if \( u \) is harmonic in \( B(x, R) \) and continuous in \( \overline{B(x, R)} \) then

\[
\frac{\partial}{\partial x_i} u(x) = \frac{n}{R^{n+1} \omega_n} \int_{|y-x|=R} (y-x)_i u(y) \, d\sigma_y
\]

b) Show that if \( z \in B(x, R) \) then

\[
|\nabla u(z)| \leq \frac{n}{R - |z|} \max_{|y|=R} |u(y)|.
\]

Hint: Apply the previous formula to balls centered at \( z \) and use the maximum principle.

c) Show that if \( u(x) \in C^2(\mathbb{R}^n) \) satisfies \( \Delta u(x) = 0 \) for every \( x \in \mathbb{R}^n \), and

\[
\lim_{|x| \to \infty} \frac{1}{|x|} \max_{|x|=R} |u(x)| = 0
\]

then \( u \) is constant. How do the functions, \( \log |x| \), if \( n = 2 \), and \( |x|^{2-n} \), if \( n > 2 \), fit into this result? Are they counter-examples?