QUALIFYING EXAMINATION

JANUARY 2000

MATH 523 - Prof. SaBarreto

1) a) Solve

$$\frac{\partial u}{\partial y} + u^5 \frac{\partial u}{\partial x} = 0,$$
$$u(x,0) = f(x).$$

b) Find the lines on the plane (x, y) where u is constant. Use this to examine whether u will develop shocks in the region $y > 0$.

2) Let $u(x,t) \in C^2(\mathbb{R}^n \times (0,\infty))$ satisfy

$$\left(\frac{\partial^2}{\partial t^2} - \Delta\right) u(x,t) = f(x,t), \quad x \in \mathbb{R}^n,\ t > 0$$
$$u(x,0) = 0, \quad \frac{\partial}{\partial t} u(x,0) = 0$$

Find a formula for $u(x,t)$ in terms of f when $n = 2, 3$.

3) Let $f \in C^\infty(\mathbb{R})$ and let $u(x,t)$ be the solution to

$$\left(\frac{\partial}{\partial t} - \Delta + f(t)\right) u(x,t) = 0, \quad x \in \mathbb{R}^n,\ t > 0,$$
$$u(x,0) = \phi(x).$$

Find a formula for u in terms of f and ϕ.

4) a) Let $\Omega \subset \mathbb{R}^n$ be an open subset and let $S \subset \Omega$ be a real analytic hypersurface. Let ϕ, ψ be real analytic functions on S. Show that there exist an open set $U \subset \Omega$, with $S \subset U$, and a unique real analytic function $u \in C^\omega(U)$ such that

$$\Delta u = 0, \quad \text{in} \quad U$$
$$u|_S = \phi, \quad \partial_\mu u|_S = \psi, \tag{1}$$

where ∂_μ is the normal derivative to S.

b) If ϕ_n, and ψ_n are sequences of real analytic functions converging uniformly to zero. Is it true that the solutions u_n of the problem (**??**) with data ϕ_n, ψ_n converge uniformly to zero?
Hint: Take $S = \{y = 0\} \subset \mathbb{R}^2$, $\phi_n(x) = 0$ and $\psi_n(x) = \frac{1}{n}\sin(nx)$.

5) Let $f \in C^2(\mathbb{R})$ satisfy $f''(t) \geq 0$ for every $t \in \mathbb{R}$ and

$$\lim_{|t|\to\infty} \frac{f(t)}{|t|} = 0.$$

Show that f is constant. What is the geometric interpretation of this result?
Hint: Suppose that $f'(t_0) > 0$ for some $t_0 \in \mathbb{R}$. Show that in this case $f(t) - f(t_0) \geq f'(t_0)(t - t_0)$ for every $t > t_0$. Prove a similar estimate if $f'(t_0) < 0$ for some t_0.

6) The Poisson kernel of the ball $B(x, R) = \{y \in \mathbb{R}^n : |x - y| < R\}$ is

$$H(x, y, \xi) = \frac{1}{R\,\omega_n} \frac{R^2 - |x - \xi|^2}{|y - \xi|^n},$$

where ω_n is the surface area on the sphere in $\mathbb{R}^n$. So, if u is harmonic in $B(x, R)$ and continuous in $\overline{B(x, R)}$, and if $|x - \xi| < R$, we have

$$u(\xi) = \frac{R^2 - |x - \xi|^2}{R\,\omega_n} \int_{|y-x|=R} \frac{1}{|y - \xi|^n} u(y)\, d\sigma_y$$

a) Use this to show that if u is harmonic in $B(x, R)$ and continuous in $\overline{B(x, R)}$ then

$$\frac{\partial}{\partial x_i} u(x) = \frac{n}{R^{n+1}\omega_n} \int_{|y-x|=R} (y - x)_i\; u(y)\, d\sigma_y$$

b) Show that if $z \in B(x, R)$ then

$$|\nabla u(z)| \leq \frac{n}{R - |z|} \max_{|y|=R} |u(y)|.$$

Hint: Apply the previous formula to balls centered at z and use the maximum principle.

c)Show that if $u(x) \in C^2(\mathbb{R}^n)$ satisfies $\Delta u(x) = 0$ for every $x \in \mathbb{R}^n$, and

$$\lim_{|x|\to\infty} \frac{1}{|x|} \max_{|x|=R} |u(x)| = 0$$

then u is constant. How do the functions, $\log |x|$, if $n = 2$, and $|x|^{2-n}$, if $n > 2$, fit into this result? Are they counter-examples?