1. (20 pts)
   (a) Prove that the function
   \[ K(x, \xi) = \frac{1}{2} |x - \xi|, \quad x \in \mathbb{R}, \xi \in \mathbb{R}, \]
   is a fundamental solution for the operator \( P = \frac{d^2}{dx^2} \) with pole \( \xi \).
   \[ \frac{d^2}{dx^2} G(x, \xi) = \delta(\xi), \quad G(0, \xi) = G(1, \xi) = 0, \quad \forall \xi \in (0, 1). \]
   Find the Green's function \( G(x, \xi) \) of the operator \( P \) above in the interval \([0, 1]\), i.e., find \( G(x, \xi) \), for \( x \in [0, 1], \xi \in (0, 1) \), such that
   \[ \frac{d^2}{dx^2} G(x, \xi) = \delta(\xi), \quad G(0, \xi) = G(1, \xi) = 0, \quad \forall \xi \in (0, 1). \]
   Plot \( G(x, \xi) \) as a function of \( x \) for a fixed \( \xi \in (0, 1) \), for example \( \xi = 1/3 \).
   (c) Using the result in (b) express the solution of the problem
   \[ u''(x) = f(x), \quad u(0) = u(1) = 0, \]
   where \( f \) is integrable over \([0, 1]\), as certain integral(s) of \( f \). Use this to solve
   \[ u'' = e^x \quad \text{in} \quad (0, 1), \quad u(0) = u(1) = 0 \quad (1) \]
   and then compare your answer with the solution of (1) that can be found directly by integration.

2. (15 pts) Solve the following initial value problem for the transport equation
   \[ \begin{cases} \partial_t u(t, x) = -v \cdot \nabla_x u(t, x) - a(x)u(t, x), & t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \\ u(0, x) = f(x), & x \in \mathbb{R}^n. \end{cases} \quad (2) \]
   Here \( 0 \neq v \in \mathbb{R}^n \) is a fixed vector, \( a(x) \) and \( f(x) \) are given continuous functions of \( x \in \mathbb{R}^n \), and \( \nabla_x = (\partial_{x_1}, \ldots, \partial_{x_n}) \).

3. (17 pts)
   (a) Find the type of the equation
   \[ yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0, \quad x \neq y \quad (3) \]
   and make a change of variables reducing (3) into its normal form.
   (b) Find the general solution of (3). Write down the solution in the original \((x, y)\) variables.
4. (16 pts) Let $K \in \mathbb{R}^n$, $n \geq 2$, be a bounded domain with smooth boundary, let $D = \mathbb{R}^n \setminus K$ be its exterior, and consider the “exterior” Dirichlet problem
\[ \Delta u = 0 \quad \text{in } D, \quad u|_{\partial D} = f, \tag{4} \]
where $f \in C(\partial D)$.
(a) Prove that the solution of (4) is unique in the class of functions $u \in C(D) \cap C^2(D)$ satisfying the limit $\lim_{|x| \to \infty} u(x) = 0$.
(b) Show that without the limit condition, the uniqueness may fail. To this end, assume that $D = \{x; |x| > 1\}$ is the complement of the unit ball in $\mathbb{R}^3$ and $f(x) = 1$, and find explicitly two different solutions of (4) in $C(D) \cap C^2(D)$ such that at least one of them does not tend to 0, as $|x| \to \infty$.

5. (15 pts) Let $D \subset \mathbb{R}^n$, $n \geq 2$ be an open set and let $u \in C^2(D)$. Prove that if for any sphere $S$ belonging to $D$ together with its interior, we have
\[ \int_S \frac{\partial u}{\partial \nu} dS = 0, \]
then $u$ is harmonic in $D$. Here $\nu$ is the outer normal to $S$ as usual.

6. (17 pts)
(a) Let $Q$ be the square $Q = \{(x,y) \in \mathbb{R}^2; 0 < x < \pi, 0 < y < \pi\}$. Solve the following heat transfer problem
\[
\begin{cases}
  u_t - \Delta u &= 0, &\text{for } t > 0, (x,y) \in Q, \\
  u|_{\partial Q} &= 0, &\text{for } t > 0, \\
  u|_{t=0} &= xy(\pi - y), &\text{for } (x,y) \in Q,
\end{cases}
\]
where $\Delta = \partial_x^2 + \partial_y^2$.
(b) Prove that $0 < u(t,x,y) < \pi^3/4$ for all $t > 0$ and $(x,y) \in Q$.
(c) Prove that there exists a constant $C > 0$ such that $0 < u(t,x,y) \leq Ce^{-2t}$ for $t \geq 0$ and $(x,y) \in Q$. 