Each problem is worth 20 points.

1. Find the solution of the problem

\[ u_y + u_x^2 z = 0 \]
\[ u(x, 0) = x^2 \]

2. For \( x^2 + y^2 < 1 \), let

\[ v(x, y) = \frac{1 - (x^2 + y^2)}{2\pi} \int_0^{2\pi} \frac{\cos \theta \sin \theta}{(x - \cos \theta)^2 + (y - \sin \theta)^2} \, d\theta \]

Find the explicit formula for \( v \) and justify your answer.
3. Let $B_1(0)$ be the unit ball in $\mathbb{R}^3$. Suppose $u \in C(B_1(0)) \cap C^2(B_1(0) \setminus \{0\})$ and $\Delta u = 0$ in $B_1(0) \setminus \{0\}$.

   a) Show that $\int_{B_1(0)} u(x) \Delta \phi(x) dx = 0$ for all $\phi \in C_0^\infty(B_1(0))$.

   b) Give an example of a function $u \in C^2(B_1(0) \setminus \{0\})$ such that $\Delta u = 0$ in $B_1(0) \setminus \{0\}$ and such that the result in a) does not hold. Compute $\int_{B_1(0)} u(x) \Delta \phi(x) dx$ for $\phi \in C_0^\infty(B_1(0))$. 


4. Let $u$ be bounded and continuous on $\mathbb{R}^n \times [0, T]$ and $u_t - \Delta u = 0$ on $\mathbb{R}^n \times (0, T)$.

   a) Show that $\sup_{\mathbb{R}^n \times [0, T]} u = \sup_{\mathbb{R}^n} u(x, 0)$.
   
   Hint: Consider $v(x, t) = u(x, t) - \varepsilon(nt + |x|^2)$.

   b) Write a formula for the solution $u$ of

   $u_t - \Delta u = 0$

   $u(x, 0) = g(x)$ where $g$ is continuous

   where $g$ is continuous with compact support in $\mathbb{R}^n$. 
5. Find a formula for the solution of

\[ u_{tt} - u_{xx} + u = 0 \quad \text{in} \quad \mathbb{R} \times (0, \infty) \]

such that

\[ u(x, 0) = f(x) \]
\[ u_t(x, 0) = g(x) \]

where \( f, g \in C^\infty_0(\mathbb{R}) \).

Hint: Let \( v(x, y, t) = h(y)u(x, t) \).

Find \( h \) so that \( v \) solves the 2-dimensional wave equation.