Each problem is worth 20 points.

1. (a) Find a solution of the Cauchy problem
\[ yu_x + xu_y = xy, \quad u = 1 \quad \text{on} \quad S := \{x^2 + y^2 = 1\}. \]
(b) Is the solution unique in a neighborhood of the point (1, 0)? Justify your answer.
2. Consider the second order PDE in \( \{ x > 0, y > 0 \} \subset \mathbb{R}^2 \)
\[
x^2 u_{xx} - y^2 u_{yy} = 0.
\]
    (a) Classify the equation and reduce it to the canonical form.
    (b) Show that the general solution of the equation is given by the formula
\[
    u(x, y) = F(xy) + \sqrt{xy} G(x/y).
\]
3. Let $\Phi$ be the fundamental solution of the Laplace equation in $\mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then the convolution

$$u(x) := (\Phi * f)(x) = \int_{\mathbb{R}^n} \Phi(x - y)f(y)dy$$

is a solution of the Poisson equation $-\Delta u = f$ in $\mathbb{R}^n$. Show that if $f$ is radial (i.e. $f(y) = f(|y|)$) and supported in $B_R := \{|x| < R\}$, then

$$u(x) = c \Phi(x), \quad \text{for any} \quad x \in \mathbb{R}^n \setminus B_R,$$

where $c = \int_{\mathbb{R}^n} f(y)dy$.

*Hint:* Use spherical (polar) coordinates and the mean value property.
4. Consider the so-called 2-dimensional wave equation with dissipation

\[
\begin{cases}
u_{tt} - \Delta u + \alpha u_t = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\
u(x, 0) = g(x), \quad u_t(x, 0) = h(x), & x \in \mathbb{R}^2,
\end{cases}
\]

where \( g, h \in C^\infty_0(\mathbb{R}^2) \) and \( \alpha \geq 0 \) is a constant.

(a) Show that for an appropriate choice of constants \( \lambda \) and \( \mu \) the function

\[v(x_1, x_2, x_3, t) := \exp(\lambda t + \mu x_3) u(x_1, x_2, t)\]

solves the 3-dimensional wave equation \( v_{tt} - \Delta v = 0 \).

(b) Use (a) to prove the following domain of dependence result: for any point \( (x_0, t_0) \in \mathbb{R}^2 \times (0, \infty) \) the value \( u(x_0, t_0) \) is uniquely determined by the values of \( g \) and \( h \) in \( B_{t_0}(x_0) := \{ x \in \mathbb{R}^2 : |x - x_0| \leq t_0 \} \).

(You may use the corresponding result for the wave equation without proof.)
5. Let $u(x,t)$ be a bounded solution of the heat equation $u_t = u_{xx}$ in $\mathbb{R} \times (0, \infty)$ with the initial condition

$$u(x,0) = u_0(x) \quad \text{for} \quad x \in \mathbb{R},$$

where $u_0 \in C^\infty(\mathbb{R})$ is $2\pi$-periodic, i.e. $u_0(x + 2\pi) = u_0(x)$. Show that

$$\lim_{t \to \infty} u(x, t) = a_0,$$

uniformly in $x \in \mathbb{R}$, where

$$a_0 := \frac{1}{2\pi} \int_0^{2\pi} u_0(x)dx.$$