QUALIFYING EXAMINATION
JANUARY 2005

MATH 523 - A. Petrosyan and P. Stefanov

Each problem is worth 20 points.

1. Find the solution to the Cauchy problem

$$yu_x - xu_y = 2xyu, \quad u|_{x=y} = x^2.$$

2. Consider the Cauchy problem

$$\begin{aligned} u_{tt} - u_{tx} - 2u_{xx} + xu_t + 5u &= 0 \\ u|_\gamma &= e^x, \\ u_t|_\gamma &= 3 - 2x, \end{aligned}$$

where γ is the curve $t = 1 - \cos x$ in the xt plane.

(a) Is there a solution of this problem near $(0,0)$? Why? What is the regularity of the solution?

(b) Find the truncated Taylor expansion of $u(x,t)$ of order 2 near the point $(0,0)$ (i.e., find the quadratic approximation u_2 of u near $(0,0)$ such that $u(x,t) = u_2(x,t) + O(|x|^3 + |t|^3)$).

3. Find an explicit solution $u(x,y)$ to the following problem

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{for } 1 < x^2 + y^2 < 4, \\ u = x & \text{for } x^2 + y^2 = 1, \\ u = 1 + xy & \text{for } x^2 + y^2 = 4. \end{cases}$$

4. Find an explicit solution to the problem

$$\begin{cases} u_t - u_{xx} = 0, & 0 < t,\ x \in \mathbf{R}, \\ u|_{t=0} = e^{3x}, & x \in \mathbf{R} \end{cases}$$

5. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary. Let $u_1 \in C^1(\bar{\Omega})$ be harmonic in Ω, and let $u_2 \in C^1(\mathbf{R}^n \setminus \Omega)$ be harmonic outside $\bar{\Omega}$. Prove that the function $u(x)$, $x \in \mathbf{R}^n$ defined by

$$u(x) = \begin{cases} u_1(x), & x \in \Omega, \\ u_2(x), & x \in \mathbf{R} \setminus \Omega \end{cases}$$

is harmonic in $\mathbf{R}^n$ if and only if

$$u_1\big|_{\partial\Omega} = u_2\big|_{\partial\Omega} \quad \text{and} \quad \partial_\nu u_1\big|_{\partial\Omega} = \partial_\nu u_2\big|_{\partial\Omega}.$$

As usual, ν denotes the exterior normal to $\partial\Omega$, that is also the interior normal to $\mathbf{R}^n \setminus \Omega$.

Hint: Prove first that u is a weak solution to $\Delta u = 0$ using Green's formula.

6. Use Hadamard's method of descent to derive the formula for the solution of initial value problem for the 1D wave equation on the whole line

$$u_{tt} = u_{xx}, \quad u|_{t=0} = f(x), \quad u_t|_{t=0} = g(x)$$

from the known formula for the solution of the 2D wave equation in the whole plane

$$u_{tt} = u_{xx} + u_{yy}.$$

Assume that f and g are as smooth as needed.