Math 523 Qualifier - August 2006
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Instructions: All work must be rigorous. Justify all steps.

1. (25 points) Consider the first order pde

   $(\sin u)u_x - u_y + uu_z = \cos u$

with Cauchy data: $u(x, y, x^3) = g(x)$,

where g is a real-valued C^∞ function defined on R.

Under what assumptions on g (if any) does the Cauchy-Kovalevskaya Theorem guarantee that there exists a unique real analytic solution u(x,y,z) in a neighborhood of the point (2,1,8)? Justify your answer. Begin with a statement of a version of the Cauchy-Kovalevskaya Theorem that is applicable here.

2. (15 points) Find an explicit solution of the Cauchy problem

   $2yu_x + u_y = 4u,$

   $u(x, 1) = x^2,$

whose domain of definition includes a neighborhood of the line y=1.

3. (15 points) Let B_R denote the interior of the open ball in R^2 with center $\vec{0}$ and radius R. Assume that u is continuous on $\overline{B_R}$ and C^2 in B_R, $\Delta u = 0$ in B_R, and

   $u(\vec{x}) = |\cos\theta|$ for $\vec{x} \in \partial B_R$,

where (r, θ) denotes the polar coordinates of $\vec{x} = (x_1, x_2)$. Compute $u(\vec{0})$ and justify your reasoning – i.e. explain why your method of computation of $u(\vec{0})$ is valid.

4. (20 points) Assume that u(x,t) is a bounded solution of the Cauchy problem

$u_{xx} - u_t = 0$ in $R \times (0, \infty)$, (*)

$u(x, 0) = arctan(x)$ for all $x \in R$, (**)

with $u \in C(R \times [0, \infty)) \bigcap C_1^2(R \times (0, \infty))$.

4.a.) Find a formula for $u_x(x, t)$ in terms of the fundamental solution of the heat equation. Prove rigorously that your formula holds.

4.b) Prove that $0 < u_x(x, t) < 1$ for all (x,t) in $R \times [0, \infty)$. State clearly any theorems from pde theory (including hypotheses and conclusions) that you use, and prove rigorously all other assertions that you make in your proof.

5. (25 points) Consider the Cauchy problem:

$u_{tt} - \Delta u = -u_t$ in R^n,

$u(x, 0) = f(x), u_t(x, 0) = g(x)$.

Assume that f and g have compact support in R^n, $f \in C^\infty(R^n)$ and $g \in C^\infty(R^n)$. Let u(x,t) be a solution in $C^2(R^n \times [0, \infty))$. Fix any $t_0 > 0$ and $x_0 \in R^n$, and define the energy:

$e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t(x, t)^2 + |\nabla u(x, t)|^2) dx$

for $0 \le t < t_0$, where $B(x_0, t_0 - t)$ denotes the open ball in R^n with center x_0 and radius $t_0 - t$, and ∇u denotes the gradient of u in the spatial components, $x_1, ..., x_n$.

5.a) Prove that $e'(t) \le 0$ for all $0 < t < t_0$.

5.b) What can you conclude from the result in 5.a) about the support of $u(\cdot, t)$ for $t > 0$ (defined as the closure of $\{x : u(x, t) \neq 0\}$)? Justify your reasoning.