Problem 1. Let $S^{n-1} \subset \mathbb{R}^n$ denote the unit sphere centered at the origin, i.e., $S^{n-1} = \{ \omega \in \mathbb{R}^n \mid |\omega| = 1 \}$, and consider the function $\phi \in C^\infty(\mathbb{R}^n)$ defined as follows

$$\phi(x) = \int_{S^{n-1}} e^{i\sqrt{\lambda} \langle x, \omega \rangle} d\sigma(\omega), \quad \lambda > 0, \quad x \in \mathbb{R}^n,$$

where $d\sigma$ denotes the $(n - 1)$-dimensional surface measure on $S^{n-1}$, and $i^2 = -1$. Prove that the function $u \in C^\infty(\mathbb{R}^{n+1})$ defined by

$$u(x,t) = e^{-\lambda t} \phi(x),$$

solves the heat equation $Hu = \Delta u - u_t = 0$ in $\mathbb{R}^{n+1}$. 

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Problem 2. Consider in the plane the solution to the Dirichlet problem

\[
\begin{cases}
\Delta u = 0 & \text{in } Q , \\
u = \phi_1 & \text{on } \partial Q ,
\end{cases}
\]

where \( Q = \{(x, y) \in \mathbb{R}^2 \mid |x| < 1, |y| < 1\} \), and \( \phi_1 \) is the function which equals 1 on one of the edges of the square \( Q \), and 0 on the remaining three edges (never mind the fact that such a \( \phi_1 \) is not continuous on \( \partial Q \), you can still uniquely solve the Dirichlet problem with such boundary datum).

a) Compute \( u(0) \).

b) Can you tell what is \( u(0) \) for the solution of an analogous Dirichlet problem for a regular polygon with \( n \) edges, and with boundary datum 1 on one edge and 0 anywhere else?

**Hint**: Use the invariance of Laplace’s operator with respect to orthogonal transformations and the maximum principle...
Problem 3. Let $n \geq 2$ and $u \in C^2(\mathbb{R}^n)$ be a solution in $\mathbb{R}^n$ of the equation $\Delta u = |x|^k$, for some $k \geq 0$. With $\sigma_{n-1}$ being the $(n-1)$-dimensional measure of the unit sphere, denote by

$$M_u(r) = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{\partial B(0,r)} u(y) d\sigma(y),$$

the spherical mean of $u$ over the sphere centered at the origin with radius $r$.

a) Prove that for every $r \geq 0$ one has

$$M_u(r) = u(0) + \frac{r^{k+2}}{(n+k)(k+2)}$$

b) Show that an estimate such as

$$|u(x)| \leq C (1 + |x|^{k+2-\epsilon}) , \quad x \in \mathbb{R}^n ,$$

is impossible for $C \geq 0$ and $\epsilon > 0$. 
**Problem 4.** A $C^1$ open set $A \subset \mathbb{R}^n$ is called *starlike* if, denoted by $\nu$ the outer unit normal to $\partial A$, one has
\[ < x, \nu(x) > \geq 0 \quad \text{for every} \quad x \in \partial A. \]
When the inequality is strict everywhere on $\partial A$, then $A$ is said *strictly starlike*.

Let $\Omega_1 \subset \overbar{\Omega_1} \subset \Omega_0$ be two connected, bounded, starlike domains, and suppose that the problem
\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega = \Omega_0 \setminus \overbar{\Omega_1}, \\
u = 1 & \text{on } \partial \Omega_1, \quad u = 0 & \text{on } \partial \Omega_0,
\end{cases}
\]
admits a solution $u \in C^3(\overbar{\Omega})$.

(i) Show that $0 < u < 1$ in $\Omega$.

(ii) Compute the equation satisfied by $v(x) \overset{\text{def}}{=} < x, \nabla u(x) >$ in $\Omega$, and use it to prove that every $C^1$ level set $E_t = \{ x \in \Omega \mid u(x) > t \}$ of $u$, $0 < t < 1$, is strictly starlike.
**Problem 5.** Let \( u \) be a solution of the initial-value problem for the wave equation

\[
\begin{cases}
  u_{tt} - \Delta u = 0 & \text{in } \mathbb{R}^3 \times [0, \infty), \\
  u(x,0) = \phi, & u_t(x,0) = \psi(x),
\end{cases}
\]

where \( \phi, \psi \in C^\infty(\mathbb{R}^3) \). For every \( t \geq 0 \) let

\[ E(t) = \sum_{|\alpha|+j \leq 2} \int_{\mathbb{R}^3} |D^\alpha_x D^j_t u(x,t)| \, dx. \]

Use Kirchhoff’s formula and an integration by parts that converts surface in solid integrals, to prove that if \( E(0) < \infty \), then one has for every \( (x,t) \in \mathbb{R}^3 \times (0, \infty) \), with \( t \geq 1 \),

\[ |u(x,t)| \leq \frac{C}{t} E(0), \]

for some constant \( C \geq 0 \) independent of \( u \).

**Hint:** On \( \partial B(x,t) \) one has \( 1 \equiv \frac{y-x}{t} \cdot \nu = \frac{y-x}{t} \cdot \nu \), where \( \nu \) is the outer unit normal on \( \partial B(x,t) \) ...
Problem 6. Let $u$ be the solution of the initial-value problem
\[
\begin{cases}
\Delta u - u_t = 0, & \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x, 0) = \phi(x), & x \in \mathbb{R}^n,
\end{cases}
\]
where $\phi \in C(\mathbb{R}^n)$ is a function such that for a fixed $R > 0$, $\phi(x) = 0$ for $|x| > R$. Prove that there exists a $C = C(n) > 0$, such that for every $x \in \mathbb{R}^n$ and $t > 0$

\[
|\nabla_x u(x, t)| \leq \frac{C}{\sqrt{t}} \left( \max_{|y| \leq R} |\phi(y)| \right).
\]

(Recall that
\[
G(x, t) = \begin{cases}
(4\pi t)^{-\frac{n}{2}} \exp \left( -\frac{|x|^2}{4t} \right), & t > 0, \\
0, & t \leq 0,
\end{cases}
\]
is the fundamental solution of the heat equation $Hu = \Delta u - u_t = 0$ in $\mathbb{R}^{n+1}$, with singularity at $(0,0)$.)