Math 523
Qualifying Examination
January 3, 2007

Name.............................................
I. D. no. ..............................................

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**Problem 1.** 1) Let \( S^{n-1} = \{ \omega \in \mathbb{R}^n \mid |\omega| = 1 \} \) be the unit sphere centered at the origin. Prove that the function \( u(x,t) = e^{i\sqrt{\lambda}t} \phi(x) \), where \( \phi \in C^\infty(\mathbb{R}^n) \) is defined by
\[
\phi(x) = \int_{S^{n-1}} e^{i\sqrt{\lambda} \langle x, \omega \rangle} \, d\sigma(\omega), \quad \lambda > 0 , \quad x \in \mathbb{R}^n,
\]
solves the wave equation \( \Box u = \Delta u - u_{tt} = 0 \) in \( \mathbb{R}^{n+1} \). Here, \( d\sigma \) denotes the \((n-1)\)-dimensional surface measure on \( S^{n-1} \), and \( i^2 = -1 \).

2) Find an explicit formula for \( u(x,t) \) when \( n = 3 \).
Problem 2. Solve the Cauchy problem for the nonlinear equation
\[
\begin{cases}
  u_t + u^2 u_x = 0 , \\
  u(x,0) = x .
\end{cases}
\]
Find the region in the \((x,t)\)-plane where the solution develops shocks (i.e., discontinuities).
Problem 3. Use Fourier transform to solve the Cauchy problem

\[ \begin{align*}
\Delta u - u - u_t &= 0, \quad \text{in } \mathbb{R}^n \times (0, \infty), \\
u(x,0) &= 1 \text{ if } |x| \leq 1, \quad u(x,0) = 0, \quad \text{if } |x| > 1.
\end{align*} \]
Problem 4. Let
\[ \Omega = \left\{ (x, y, z) \in \mathbb{R}^3 \mid \frac{x^2}{36} + \frac{y^2}{9} + \frac{z^2}{25} < 1 \right\}, \]
and assume that \( u \in C^2(\Omega) \) solves the problem
\[
\begin{aligned}
\Delta u &= -1, \quad \text{in } \Omega, \\
u &= 0, \quad \text{on } \partial\Omega.
\end{aligned}
\]
Prove that
\[ \frac{3}{2} \leq u(0,0,0) \leq 6. \]
Problem 5. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of class $C^1$, and $b \in C^1(\mathbb{R}^{n+1}; \mathbb{R}^n)$. Prove that there exists a constant $C = C(\Omega, n) > 0$ sufficiently small such that, if $\|(\text{div } b)^-\|_{L^\infty(\mathbb{R}^{n+1})} \leq C,$ then there exists at most one solution $u \in C^{2,1}(\overline{\Omega} \times [0, \infty))$ to the problem
\[
\begin{cases}
\Delta u + \langle b, Du \rangle - u_t = 0 & \text{in } \Omega \times [0, \infty), \\
u(x, 0) = \phi(x), & x \in \Omega, \\
u(x, t) = \psi(x, t), & (x, t) \in \partial\Omega \times (0, \infty).
\end{cases}
\]
Here, for a given $a \in \mathbb{R}$, $a^+, a^-$ respectively denote its positive and negative part, so that $a = a^+ - a^-$. 

Note: You can assume the validity of the Poincaré inequality
\[
\int_{\Omega} f(x)^2 \, dx \leq C^*(n)(\text{diam } \Omega)^2 \int_{\Omega} |Df|^2 \, dx,
\]
where $C^*(n) > 0$ depends only on the dimension $n$, and $f$ is an arbitrary function in $C^1(\overline{\Omega})$ such that $f = 0$ on $\partial\Omega$. 