Math 523  
Qualifying Examination  
August, 2008  
Prof. N. Garofalo

Name.............................................  
I. D. no. ..........................................  

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Problem 1. 1) Let $\Omega \subset \mathbb{R}^n$ be an open set and consider a sequence $\{f_k\}_{k \in \mathbb{N}}, f_k \in C^2(\Omega)$, of harmonic functions in $\Omega$ such that $0 \leq f_k \leq f_{k+1}$ and for which

$$f(x) \overset{\text{def}}{=} \sup_{k \in \mathbb{N}} f_k(x) < \infty, \quad \text{for every } x \in \Omega.$$

Prove that $f$ is harmonic in $\Omega$. 
Problem 2. Let $u$ be a solution of the initial value problem for the nonlinear equation

$$\begin{align*}
    u_t + uu_x &= 0 , \\
    u(x,0) &= x .
\end{align*}$$

Find the region in the $(x,t)$-plane where the solution develops shocks (i.e., discontinuities).
Problem 3. Use Fourier transform to solve the Cauchy problem
\[
\begin{aligned}
&u_{xx} - u_{tt} = 0, \quad \text{in } \mathbb{R} \times (0, \infty), \\
u_t(x, 0) = 1 \text{ if } |x| \leq 1, \quad u_t(x, 0) = 0 \text{ if } |x| > 1, \quad u(x, 0) = 0.
\end{aligned}
\]
Problem 4. (i) Let $\mathbb{S}^{n-1} = \{ \omega \in \mathbb{R}^n \mid |\omega| = 1 \}$ be the unit sphere centered at the origin, and define

$$
\phi(x) = \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda} \cdot \langle x, \omega \rangle} \, d\sigma(\omega), \quad \lambda > 0, \quad x \in \mathbb{R}^n,
$$

where $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\mathbb{S}^{n-1}$, and $i^2 = -1$. Prove that the function $u(x,t) = e^{-\lambda t} \phi(x)$, solves the heat equation $Hu = \Delta u - u_t = 0$ in $\mathbb{R}^{n+1}$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and suppose that $u \in C^3(\Omega \times (0,\infty))$ be a solution to $Hu = \Delta u - u_t = 0$ in $\Omega \times (0,\infty)$. Prove that the function $f = |Du|^2 + u_t^2$ cannot attain a maximum at a point $(x_0,t_0)$, with $x_0 \in \Omega$ and $t_0 > 0$, unless $u \equiv \text{constant}$ in $\Omega \times (0,t_0)$. 
Problem 5. For $x = (x_1,...,x_n) \in \mathbb{R}^n$ and a given function $\phi$, indicate $\phi_{x_i} = \frac{\partial \phi}{\partial x_i}, i = 1,...,n$.

Solve the non-homogeneous initial value problem

$$\begin{cases}
\phi_t + \phi_{x_1} - \phi_{x_n} = 2t + |x|^2, & \text{in } \mathbb{R}^n \times (0, \infty), \\
\phi(x, 0) = x_1^2 - x_n^2, & x \in \mathbb{R}^n.
\end{cases}$$