MA–523 Qualifying Exam, January 2008

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Each problem is worth 20 points. Everywhere in this exam, Ω be a bounded domain (an open connected set) in $\mathbf{R}^n$ with smooth boundary.

1. Let u be a harmonic function in Ω, continuous in $\bar{\Omega}$.

(a) Show that if

$$|u(x)| \le \sum_{|\alpha|\le 1} C_\alpha x^\alpha, \quad \text{for all } x \in \partial\Omega,$$

and for some constants C_α, then the same inequality holds inside Ω, as well (with the same constants).

(b) Show that the inequality

$$|u(x)| \le \sum_{|\alpha|\le 2} C_\alpha x^\alpha, \quad \text{for all } x \in \partial\Omega, \tag{1}$$

does not necessarily imply that the same inequality (with the same constants) holds inside Ω, as well. In other words, show that for some choice of $\{C_\alpha\}_{|\alpha|\le 2}$, the inequality (1) does not imply the same inequality in Ω.

2. (a) Prove that there is no more than one solution $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ to the Laplace equation with Robin boundary conditions

$$\begin{aligned} -\Delta u &= f \qquad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \alpha u &= h \quad \text{ on } \partial\Omega, \end{aligned}$$

where f and h are given functions, ν is the outer normal to $\partial\Omega$, and $\alpha > 0$ is a given function.

(b) Let $\Omega = B(0,1)$. Show that if $\alpha = -1$, then there is no uniqueness.

3. Let $u \in C^2(\bar{\Omega} \times [0,\infty))$ solve the following initial boundary value problem for the wave equation with an absorption term

$$\begin{cases} u_{tt} + u_t &= \Delta u \quad \text{for } x \in \Omega,\ t > 0, \\ \partial u/\partial \nu &= 0 \quad \text{for } x \in \partial\Omega,\ t \ge 0, \\ u &= f(x) \quad \text{for } t = 0, \\ u_t &= g(x) \quad \text{for } t = 0. \end{cases}$$

Here f and g are smooth functions with compact support, and ν is the unit outward normal.

(a) Let

$$E(t) = \frac{1}{2}\int_\Omega \left(|u_t|^2 + |\nabla_x u|^2\right) dx$$

be the energy. Prove that

$$E(t) \le E(0) \quad \text{for any } t \ge 0.$$

(b) Is there uniqueness of the solution (in the class $u \in C^2(\bar{\Omega} \times [0,\infty))$)? Explain.

(c) Solve the following one-dimensional version of this problem

$$\begin{cases} u_{tt} + u_t = u_{xx} & \text{for } 0 < x < \pi,\ t > 0, \\ u_x(0,t) = u_x(\pi,t) = 0 & \text{for } t \geq 0, \\ u = \cos x & \text{for } t = 0, \\ u_t = 0 & \text{for } t = 0. \end{cases}$$

4. Consider the equation

$$4u_{xx} - u_{yy} - 8u_x + 4u_y + 32 = 0. \tag{2}$$

(a) Find the characteristic curves of the equation and perform a characteristic change of variables that would reduce that equation to its canonical form.

(b) Using (a), find the general solution of (2).

(c) Find all points P on the parabola $\gamma := \{(x,y);\ y = x^2\}$ with the property that there exists a solution near P solving (1) and satisfying the conditions

$$u|_\gamma = \sin(x+y) \quad u_y|_\gamma = xy^2.$$

State clearly the theorem that you use.

5. Solve the problem

$$\begin{aligned} u_x + (2x+1)u_y + u^2 &= 0, \quad x > 0,\ y > 0, \\ u(x,0) &= 0, \quad x > 0, \\ u(0,y) &= y, \quad y > 0. \end{aligned}$$

Show that the solution u to the problem can be extended to the whole first quadrant $x \geq 0$, $y \geq 0$.