1. Consider the following Cauchy problem:
\[ xu_x - yu_y = u - 1, \]
\[ u(x, x) = 1 + x^3. \]

   a) (8 pts.) At what values \( x_0 \) is there a unique \( C^1 \) solution in a neighborhood of \((x_0, x_0)\)? Cite a theorem to support your answer.

   b) (12 pts.) Find the solution near the points \((x_0, x_0)\) found in 1.a).

2.) (20 pts.) Let \( D \) be a bounded smooth domain in \( \mathbb{R}^n \). Assume that \( u \) is a given function in \( C^3(D) \cap C^2(\overline{D}) \), and \( \Delta u = 0 \) in \( D \). Can \( (\partial u / \partial x_1)^2 \) have an interior maximum in \( D \)? Justify your answer.

3.a.) (10 pts) Let \( \Omega \) be a bounded \( C^2 \) domain in \( \mathbb{R}^n \). Assume:
\( u, u_t, u_{x_i}, u_{x_i x_j} \) are in \( C(\overline{\Omega} \times [0, \infty)) \) for all \( 1 \leq i, j \leq n \),
\( u_t - \Delta u = 0 \) in \( \Omega \times [0, \infty) \),
and \( u = 0 \) in \( \partial \Omega \times [0, \infty) \).

Show that for each \( T > 0 \):
\[ (*) \int_{\Omega} u^2(x, T) dx \leq \int_{\Omega} u^2(x, 0) dx. \]

Hint: \( 0 = 2u(u_t - \Delta u) \) in \( \Omega \times (0, T) \). Integrate by parts.

3.b.) (10 pts.) By integrating by parts as in 3.a) on \( B_R(0) \times (0, T) \) and letting \( R \to \infty \), prove that if \( v \) is a continuous, bounded solution in \( \mathbb{R}^n \times [0, \infty) \) of:
\( v_t - \Delta v = 0 \) in \( \mathbb{R}^n \times [0, \infty) \),
\( v(x, 0) = f(x) \) for all \( x \) in \( \mathbb{R}^n \),
\( \int_{\mathbb{R}^n} |f(x)|^2 dx < \infty \),
where \( f \) is \( C^\infty \) with compact support in \( \mathbb{R}^n \),
and \( v_t, v_{x_i}, v_{x_i x_j} \) are in \( C(\mathbb{R}^n \times [0, \infty)) \) for all \( 1 \leq i, j \leq n \),
then \( \int_{\mathbb{R}^n} |v(x, T)|^2 dx \leq \int_{\mathbb{R}^n} |v(x, 0)|^2 dx. \)
4.) Consider the solution of
\[ u_{tt} - \Delta u = 0 \] in \( \mathbb{R}^3 \times (0, \infty) \),
\[ u(x, 0) = 0 \text{ and } u_t(x, 0) = g(x) \] for all \( x \) in \( \mathbb{R}^3 \),

where \( u \) is in \( C^2(\mathbb{R}^3 \times [0, \infty)) \). Assume \( g \) is \( C^\infty \) with compact support in \( \mathbb{R}^3 \) and 
\[ g(x) > 0 \text{ when } |x| < 1, \ g(x) = 0 \text{ when } |x| \geq 1. \]

(a.) (10 pts.) What is the solution to the above problem?

(b.) (10 pts.) For each \( x_0 \) in \( \mathbb{R}^3 \), identify \( Z(x_0) \equiv \{ t > 0 : u(x_0, t) = 0 \} \). Justify your answer.

5. Let \( \Gamma = \{(x, y) \in \mathbb{R}^2 : y = x^2 \} \). Consider the Cauchy problem:
\[ 4yu_{yy} - 4xu_{xy} + 3xy^2u_{xx} = 0, \]
\[ u(x, y) = x^3y^2 - 2y \] on \( \Gamma \),
\[ u_y(x, y) = 3xy^2 \] on \( \Gamma \).

(a.) (10 pts.) At what points \( (x_0, y_0) \) on \( \Gamma \) is there a real analytic solution of this problem in a neighborhood of \( (x_0, y_0) \)? Cite a theorem to justify your answer.

(b.) (10 pts.) Compute the terms of order \( \leq 1 \) in the power series for the solution expanded about the point \( (1,1) \).