Math 523
Qualifying Examination
August 7, 2012
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		30
3		20
4		30
5		20
Total		120

Problem 1. Let ϕ be a continuous function on $\mathbb{R}^n$ with compact support, $F \in C(\mathbb{R}^n \times (0,\infty))$ with compact support. Write an explicit formula for the solution of the non-homogeneous Cauchy problem

$$\begin{cases} u_{x_1} - 3u_{x_n} + u_t = F, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases}$$

Here, for $i = 1, ..., n$, we have set $u_{x_i} = \frac{\partial u}{\partial x_i}$.

Problem 2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

a) The Fourier transform of the function $f(x) = e^{-<Ax,x>}$, is

A. $\hat{f}(\xi, \eta, \zeta) = \frac{2\pi^{\frac{3}{2}}}{\sqrt{6}} e^{-\pi^2(\xi^2+2\eta^2+3\zeta^2)}$

B. $\hat{f}(\xi, \eta, \zeta) = \frac{\pi^{\frac{3}{2}}}{\sqrt{6}} e^{-\frac{\pi^2}{\sqrt{6}}(\xi^2+2\eta^2+3\zeta^2)}$

C. $\hat{f}(\xi, \eta, \zeta) = \frac{\pi^{\frac{3}{2}}}{6} e^{-\pi^2(\xi^2+2\eta^2+3\zeta^2)}$

D. $\hat{f}(\xi, \eta, \zeta) = \frac{\pi^{\frac{3}{2}}}{\sqrt{6}} e^{-\pi^2(\xi^2+\frac{\eta^2}{2}+\frac{\zeta^2}{3})}$

E. $\hat{f}(\xi, \eta, \zeta) = \frac{\pi^{\frac{3}{2}}}{6} e^{-\pi^2(\xi^2+\frac{\eta^2}{2}+\frac{\zeta^2}{3})}$

b) Using part a) solve the Cauchy problem

$$\begin{cases} \operatorname{div}(A\nabla u) - u_t = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^3, \end{cases}$$

where ϕ is a continuous function on $\mathbb{R}^3$ with compact support.

Problem 3. Let $f \in C^2(\mathbb{R}^n)$ be a solution of $\Delta f = |x|^\alpha$, for some given $\alpha > 0$. Let $M_f(r) = \frac{1}{\sigma_{n-1} r^{n-1}} \int_{S(r)} f(x) d\sigma(x)$, be the spherical mean of f over the sphere $S(r) = \{x \in \mathbb{R}^n \mid |x| = r\}$.

1) Prove that

$$M_f(r) = f(0) + \frac{r^{\alpha+2}}{(\alpha+2)(\alpha+n)}, \qquad r > 0.$$

2) Prove that there cannot exist $C \geq 0$ and $0 < \varepsilon < \alpha + 2$ such that

$$|f(x)| \leq C(1 + |x|)^\varepsilon, \qquad \forall x \in \mathbb{R}^n.$$

Problem 4. Let $B_R = \{x \in \mathbb{R}^n \mid |x| < R\}$.

a) Prove that if $f \in C^2(B_R)$ is harmonic in B_R and spherically symmetric (i.e., $f(Tx) = f(x)$ for every $T \in \mathbb{O}(n)$ and for every $x \in B_R$), then f must be constant.

b) Is the same conclusion necessarily true if $f \in C^2(B_R \setminus \{0\})$ is harmonic in $B_R \setminus \{0\}$ and spherically symmetric there?

Problem 5.

1) Let $\mathbb{S}^{n-1} = \{\omega \in \mathbb{R}^n \mid |\omega| = 1\}$ be the unit sphere centered at the origin. Prove that the function $u(x,t) = e^{i\sqrt{\lambda}t}\phi(x)$, where $\phi \in C^\infty(\mathbb{R}^n)$ is defined by

$$\phi(x) \;=\; \int_{\mathbb{S}^{n-1}} e^{i\sqrt{\lambda}\ <x,\omega>}\, d\sigma(\omega)\ , \qquad \lambda > 0\ , \quad x \in \mathbb{R}^n\ ,$$

solves the wave equation $\Box u = \Delta u - u_{tt} = 0$ in $\mathbb{R}^{n+1}$. Here, $d\sigma$ denotes the $(n-1)$-dimensional surface measure on $\mathbb{S}^{n-1}$, and $i^2 = -1$.

2) Find an explicit formula for $u(x,t)$ when $n = 3$.