Math 523
Qualifying Examination
January, 2012
Prof. N. Garofalo

Name..

I. D. no. ....

Problem	Score	Max. pts.
1		20
2		30
3		20
4		30
5		20
Total		120

Problem 1. Let ϕ be a continuous function on $\mathbb{R}^n$ with compact support, $F \in C(\mathbb{R}^n \times (0,\infty))$ with compact support, $a \in \mathbb{R}^n \setminus \{0\}$. Write an explicit formula for the solution of the non-homogeneous Cauchy problem

$$\begin{cases} < a, \nabla u > + u_t = F, & \text{in } \mathbb{R}^n \times (0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^n. \end{cases}$$

Problem 2. Consider the matrix $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

a) The Fourier transform of the function $f(x) = e^{-<Ax,x>}$, is

A. $\hat{f}(\xi,\eta,\zeta) = \frac{2\pi^{\frac{3}{2}}}{\sqrt{6}} e^{-\pi^2(2\xi^2+3\eta^2+\zeta^2)}$

B. $\hat{f}(\xi,\eta,\zeta) = \frac{\pi^{\frac{5}{2}}}{\sqrt{6}} e^{-\frac{\pi^2}{\sqrt{6}}(\xi^2+3\eta^2+2\zeta^2)}$

C. $\hat{f}(\xi,\eta,\zeta) = \frac{\pi^{\frac{3}{2}}}{6} e^{-\pi^2(2\xi^2+3\eta^2+\zeta^2)}$

D. $\hat{f}(\xi,\eta,\zeta) = \frac{\pi^{\frac{3}{2}}}{\sqrt{6}} e^{-\pi^2(\frac{\xi^2}{2}+\frac{\eta^2}{3}+\zeta^2)}$

E. $\hat{f}(\xi,\eta,\zeta) = \frac{\pi^{\frac{3}{2}}}{6} e^{-\pi^2(\frac{\xi^2}{2}+\frac{\eta^2}{3}+\zeta^2)}$

b) Using part a) solve the Cauchy problem

$$\begin{cases} \operatorname{div}(A\nabla u) - u_t = 0 & \text{in } \mathbb{R}^3 \times (0,\infty), \\ u(x,0) = \phi(x), & x \in \mathbb{R}^3, \end{cases}$$

where ϕ is a continuous function on $\mathbb{R}^3$ with compact support.

Problem 3. Let $\Omega \subset \mathbb{R}^2$ be the open set defined by

$$\Omega = \Big((-4,4) \times (-4,4) \Big) \setminus \Big([-1,1] \times [-1,1] \Big).$$

Suppose that $u \in C^{2,1}(\Omega)$, $u \leq 0$, be such that

$$u_{xx} - u_t \geq 0, \qquad \text{in } \Omega.$$

If $u(2,0) = 0$, determine the subset of Ω (possibly constituted only of the point $(2,0)$) in which the function u vanishes. Answers in the form a (correct) picture will be accepted.

Problem 4. Let $n \geq 2$, and consider the inversion $\Phi : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\}$ given by $\Phi(x) = \frac{x}{|x|^2}$.

a) Prove that, given $a > 0$, Φ maps the half-space $H_a^+ = \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > a\}$ onto the ball $B((0, \frac{1}{2a}), \frac{1}{2a}) \subset \mathbb{R}^n$ centered at the point $(0, \frac{1}{2a})$ (here $0 \in \mathbb{R}^{n-1}$), and with radius $\frac{1}{2a}$.

b) Let $\tilde{f}(x) = |x|^{2-n} f(\frac{x}{|x|^2})$ be the so-called *Kelvin transform* of a function f. Compute the Kelvin transform of the function $f(x) = x_n - a$ on H_a^+.

Problem 5.

a) Prove that for every $\alpha > 0$ the function
$$f(x) = \int_{\mathbb{S}^{n-1}} e^{-i\alpha <x,\omega>} d\sigma(\omega),$$
solves the equation $\Delta f + \alpha^2 f = 0$ in $\mathbb{R}^n$.

b) Use part a) and separation of variables, to find an explicit (formal) solution to the Cauchy problem
$$\begin{cases} \Delta u - u_{tt} = 0 & \text{in} \quad \mathbb{R}^n \times (0,\infty), \\ u(x,0) = f(x), \quad u_t(x,0) = 0 & x \in \mathbb{R}^n. \end{cases}$$