MA52300 Qualifying Examination

January 2014 — Prof. Petrosyan

1. Consider the first order equation in $\mathbb{R}^2$ [20pt]

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$

(a) Find the characteristic curves of the equation.

(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on f so that the problem is solvable in a neighborhood of the point $(1, 0)$.

2. Let u be a continuous bounded solution of the initial value problem for the Laplace equation [20pt]

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\} \\ u(x', 0) = g(x') & \text{for } x' \in \mathbb{R}^{n-1}, \end{cases}$$

where g is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$$u(x) \to 0 \quad \text{as } |x| \to \infty$$

for $x \in \mathbb{R}^n_+$.

3. Let u be a bounded solution of the heat equation [20pt]

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

with the initial conditions $u(x,0) = g(x)$, where g is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x-y|^\alpha, \quad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0,1]$. Show that

$$|u(x,t) - u(y,t)| \leq M|x-y|^\alpha, \quad x, y \in \mathbb{R},\ t > 0, \quad \text{and}$$
$$|u(x,t) - u(x,s)| \leq C_\alpha M|t-s|^{\alpha/2}, \quad x \in \mathbb{R},\ t, s > 0.$$

[Hint: For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t-s|}$.]

4. Let u be a positive harmonic function in the unit ball B_1 in $\mathbb{R}^n$. Show that [20pt]

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant M depending only on the dimension n.

[Hint: Use the interior derivative estimate $|Du(x)| \leq \frac{C_n}{r} \sup_{B_r(x)} |u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions].

5. Let u be a C^2 solution of the initial value problem [20pt]

$$\begin{aligned} u_{tt} - \Delta u = |x|^k \quad & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = 0, \quad u_t = 0 \quad & \text{on } \mathbb{R}^n \times \{0\}. \end{aligned}$$

for some $k \geq 0$. Prove that there exists a function $\phi(r)$ such that

$$u(x,t) = t^{k+2}\phi(|x|/t).$$

[Hint: As one of the steps show that u is $(k+2)$-homogeneous in (x,t) variables, i.e. $u(\lambda x, \lambda t) = \lambda^{k+2}u(x,t)$ for any $\lambda > 0$.]