1. Consider the first order equation in $\mathbb{R}^2$

$$x_2 u_{x_1} + x_1 u_{x_2} = 0.$$ 

(a) Find the characteristic curves of the equation.

(b) Consider the Cauchy problem for this equation prescribed on the line $x_1 = 1$:

$$u(1, x_2) = f(x_2).$$

Find a necessary condition on $f$ so that the problem is solvable in a neighborhood of the point $(1, 0)$. 
2. Let $u$ be a continuous bounded solution of the initial value problem for the Laplace equation

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \mathbb{R}^n_+ = \{(x', x_n) \in \mathbb{R}^n : x_n > 0\} \\
u(x', 0) &= g(x') \quad \text{for } x' \in \mathbb{R}^{n-1},
\end{align*}
\]

where $g$ is a continuous function with compact support in $\mathbb{R}^{n-1}$. Here $n \geq 2$. Prove that

$u(x) \to 0$ as $|x| \to \infty$

for $x \in \mathbb{R}^n_+$. 
3. Let $u$ be a bounded solution of the heat equation

$$\Delta u - u_t = 0 \quad \text{in } \mathbb{R} \times (0, \infty)$$

with the initial conditions $u(x,0) = g(x)$, where $g$ is a bounded continuous function on $\mathbb{R}$ satisfying the Hölder condition

$$|g(x) - g(y)| \leq M|x - y|^{\alpha}, \quad x, y \in \mathbb{R}$$

with a constant $\alpha \in (0, 1]$. Show that

$$|u(x,t) - u(y,t)| \leq M|x - y|^{\alpha}, \quad x, y \in \mathbb{R}, \quad t > 0,$$

and

$$|u(x,t) - u(x,s)| \leq C\alpha M|t - s|^{\alpha/2}, \quad x \in \mathbb{R}, \quad t, s > 0.$$

[Hint: For the last inequality, in the representation formula of $u(x,t)$ as a convolution with the heat kernel $\Phi(y,t)$, make a change of variables $z = y/\sqrt{t}$ and use that $|\sqrt{t} - \sqrt{s}| \leq \sqrt{|t - s|}$.]

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4. Let $u$ be a positive harmonic function in the unit ball $B_1$ in $\mathbb{R}^n$. Show that

$$|D(\ln u)| \leq M \quad \text{in } B_{1/2}$$

for a constant $M$ depending only on the dimension $n$.

[Hint: Use the interior derivative estimate $|Du(x)| \leq \frac{C_n}{r} \sup_{B_r(x)} |u|$ for $B_r(x) \subset B_1$ as well as the Harnack inequality for harmonic functions].
Let $u$ be a $C^2$ solution of the initial value problem

$$
\begin{align*}
    u_{tt} - \Delta u &= |x|^k \quad \text{in } \mathbb{R}^n \times (0, \infty) \\
    u &= 0, \quad u_t = 0 \quad \text{on } \mathbb{R}^n \times \{0\}.
\end{align*}
$$

for some $k \geq 0$. Prove that there exists a function $\phi(r)$ such that

$$
    u(x, t) = t^{k+2} \phi(|x|/t).
$$

[Hint: As one of the steps show that $u$ is $(k+2)$-homogeneous in $(x, t)$ variables, i.e. $u(\lambda x, \lambda t) = \lambda^{k+2} u(x, t)$ for any $\lambda > 0$.]